1.6

3 A different A produces the circulant second-difference matrix $C = A^{\mathrm{T}}A$:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \quad \text{gives} \quad A^{\mathrm{T}}A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

How can you tell from A that $C = A^{\mathrm{T}}A$ is only semidefinite? Which vectors solve $Au = 0$ and therefore $Cu = 0$? Note that chol(C) will fail.

4 Confirm that the circulant $C = A^{\mathrm{T}}A$ above is semidefinite by the pivot test. Write $u^{\mathrm{T}}Cu$ as a sum of *two squares* with the pivots as coefficients. (The eigenvalues $0, 3, 3$ give another proof that C is semidefinite.)

5 $u^{\mathrm{T}}Cu \geq 0$ means that $u_1^2 + u_2^2 + u_3^2 \geq u_1u_2 + u_2u_3 + u_3u_1$ for any u_1, u_2, u_3. A more unusual way to check this is by the Schwarz inequality $|v^{\mathrm{T}}w| \leq \|v\|\,\|w\|$:

$$|u_1u_2 + u_2u_3 + u_3u_1| \leq \sqrt{u_1^2 + u_2^2 + u_3^2}\,\sqrt{u_2^2 + u_3^2 + u_1^2}\,.$$

Which u's give *equality*? Check that $u^{\mathrm{T}}Cu = 0$ for those u.

6 For what range of numbers b is this matrix positive definite?

$$K = \begin{bmatrix} 1 & b \\ b & 4 \end{bmatrix}.$$

There are two borderline values of b when K is only semidefinite. In those cases write $u^{\mathrm{T}}Ku$ with only one square. Find the pivots if $b = 5$.

7 Is $K = A^{\mathrm{T}}A$ or $M = B^{\mathrm{T}}B$ positive definite (independent columns in A or B)?

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

We know that $u^{\mathrm{T}}Mu = (Bu)^{\mathrm{T}}(Bu) = (u_1 + 4u_2)^2 + (2u_1 + 5u_2)^2 + (3u_1 + 6u_2)^2$. Show how the three squares for $u^{\mathrm{T}}Ku = (Au)^{\mathrm{T}}(Au)$ collapse into one square.

Problems 8–16 are about tests for positive definiteness.

8 Which of A_1, A_2, A_3, A_4 has two positive eigenvalues? Use the tests $a > 0$ and $ac > b^2$, don't compute the λ's. Find a vector u so that $u^{\mathrm{T}}A_1u < 0$.

$$A_1 = \begin{bmatrix} 5 & 6 \\ 6 & 7 \end{bmatrix} \quad A_2 = \begin{bmatrix} -1 & -2 \\ -2 & -5 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & 10 \\ 10 & 100 \end{bmatrix} \quad A_4 = \begin{bmatrix} 1 & 10 \\ 10 & 101 \end{bmatrix}.$$

9 For which numbers b and c are these matrices positive definite?

$$A = \begin{bmatrix} 1 & b \\ b & 9 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & 4 \\ 4 & c \end{bmatrix}.$$

With the pivots in D and multiplier in L, factor each A into LDL^{T}.

2.3

10 Show that $f(x, y) = x^2 + 4xy + 3y^2$ does not have a minimum at $(0, 0)$ even though it has positive coefficients. Write f as a *difference* of squares and find a point (x, y) where f is negative.

11 The function $f(x, y) = 2xy$ certainly has a saddle point and not a minimum at $(0, 0)$. What symmetric matrix S produces this f? What are its eigenvalues?

12 Test the columns of A to see if $A^{\mathrm{T}}A$ will be positive definite in each case:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}.$$

13 Find the 3 by 3 matrix S and its pivots, rank, eigenvalues, and determinant:

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} & & \\ & S & \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 4(x_1 - x_2 + 2x_3)^2.$$

14 Which 3 by 3 symmetric matrices S produce these functions $f = x^{\mathrm{T}}Sx$? Why is the first matrix positive definite but not the second one?

(a) $f = 2(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3)$

(b) $f = 2(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_1x_3 - x_2x_3)$.

15 For what numbers c and d are A and B positive definite? Test the three upper left determinants (1 by 1, 2 by 2, 3 by 3) of each matrix:

$$A = \begin{bmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & d & 4 \\ 3 & 4 & 5 \end{bmatrix}.$$

16 *If A is positive definite then A^{-1} is positive definite.* Best proof: The eigenvalues of A^{-1} are positive because ____. *Second proof* (only quick for 2 by 2):

The entries of $A^{-1} = \dfrac{1}{ac - b^2} \begin{bmatrix} c & -b \\ -b & a \end{bmatrix}$ pass the determinant tests ____.

17 A positive definite matrix cannot have a zero (or even worse, a negative number) on its diagonal. Show that this matrix fails to have $u^{\mathrm{T}}Au > 0$.

$$\begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} 4 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \text{ is not positive when } (u_1, u_2, u_3) = (\ ,\ ,\ ).$$

18 A diagonal entry a_{jj} of a symmetric matrix cannot be smaller than all the λ's. If it were, then $A - a_{jj}I$ would have ____ eigenvalues and would be positive definite. But $A - a_{jj}I$ has a zero on the main diagonal.

19 If all $\lambda > 0$, show that $u^{\mathrm{T}}Ku > 0$ for *every* $u \neq 0$, not just the eigenvectors x_i. Write u as a combination of eigenvectors. Why are all "cross terms" $x_i^{\mathrm{T}}x_j = 0$?

$$u^{\mathrm{T}}Ku = (c_1x_1+\cdots+c_nx_n)^{\mathrm{T}}(c_1\lambda_1x_1+\cdots+c_n\lambda_nx_n) = c_1^2\lambda_1x_1^{\mathrm{T}}x_1+\cdots+c_n^2\lambda_nx_n^{\mathrm{T}}x_n > 0$$

20 Without multiplying $A = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}2 & 0\\ 0 & 5\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$, find

(a) the determinant of A (b) the eigenvalues of A
(c) the eigenvectors of A (d) a reason why A is symmetric positive definite.

21 For $f_1(x,y) = \frac{1}{4}x^4+x^2y+y^2$ and $f_2(x,y) = x^3+xy-x$ find the second derivative (Hessian) matrices H_1 and H_2:

$$H = \begin{bmatrix}\partial^2 f/\partial x^2 & \partial^2 f/\partial x\partial y\\ \partial^2 f/\partial y\partial x & \partial^2 f/\partial y^2\end{bmatrix}.$$

H_1 is positive definite so f_1 is concave up (= convex). Find the minimum point of f_1 and the saddle point of f_2 (look where first derivatives are zero).

22 The graph of $z = x^2 + y^2$ is a bowl opening upward. *The graph of $z = x^2 - y^2$ is a saddle.* The graph of $z = -x^2 - y^2$ is a bowl opening downward. What is a test on a, b, c for $z = ax^2 + 2bxy + cy^2$ to have a saddle at $(0,0)$?

23 Which values of c give a bowl and which give a saddle point for the graph of $z = 4x^2 + 12xy + cy^2$? Describe this graph at the borderline value of c.

24 Here is another way to work with the quadratic function $P(u)$. Check that

$$P(u) = \frac{1}{2}u^{\mathrm{T}}Ku - u^{\mathrm{T}}f \quad \text{equals} \quad \frac{1}{2}(u - K^{-1}f)^{\mathrm{T}}K(u - K^{-1}f) - \frac{1}{2}f^{\mathrm{T}}K^{-1}f.$$

The last term $-\frac{1}{2}f^{\mathrm{T}}K^{-1}f$ is P_{min}. The other (long) term on the right side is always ____. When $u = K^{-1}f$, this long term is zero so $P = P_{\mathrm{min}}$.

25 Find the first derivatives in $f = \partial P/\partial u$ and the second derivatives in the matrix H for $P(u) = u_1^2+u_2^2-c(u_1^2+u_2^2)^4$. Start Newton's iteration (21) at $u^0 = (1,0)$. Which values of c give a next vector u^1 that is closer to the local minimum at $u^* = (0,0)$? Why is $(0,0)$ not a global minimum?

26 Guess the smallest 2, 2 block that makes $[\boldsymbol{C^{-1}}\ \ \boldsymbol{A};\ \ \boldsymbol{A^{\mathrm{T}}}\ \ \_\_\_\_]$ semidefinite.

27 If H and K are positive definite, explain why $M = \begin{bmatrix}H & 0\\ 0 & K\end{bmatrix}$ is positive definite but $N = \begin{bmatrix}K & K\\ K & K\end{bmatrix}$ is not. Connect the pivots and eigenvalues of M and N to the pivots and eigenvalues of H and K. How is chol(M) constructed from chol(H) and chol(K)?

2.1

3 Find $(A^TCA)^{-1}$ in the fixed-free example by multiplying $A^{-1}C^{-1}(A^T)^{-1}$. Check the special case with all $c_i = 1$ and $C = I$.

4 In the free-free case when A^TCA in (11) is singular, add the three equations $A^TCAu = f$ to show that we need $f_1+f_2+f_3 = 0$. Find a solution to $A^TCAu = f$ when the forces $f = (-1, 0, 1)$ balance themselves. Find all solutions!

5 In the fixed-fixed case, what are the reaction forces on the top of spring 1 and on the bottom of spring 4? They should balance the total force $3mg$ from gravity, pulling down on the three masses.

6 With $c_1 = c_3 = 1$ in the fixed-free case, suppose you strengthen spring 2. Find $K = A^TCA$ for $c_2 = 10$ and $c_2 = 100$. Compute $u = K^{-1}f$ with equal masses $f = (1, 1, 1)$.

7 With $c_1 = c_3 = c_4 = 1$ in the fixed-fixed case, weaken spring 2 in the limit to $c_2 = 0$. Does $K = A^TCA$ remain invertible? Solve $Ku = f = (1, 1, 1)$ and explain the answer physically.

8 For one free-free spring only, show that $K = c\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ = "element matrix."

(a) Assemble K's for springs 2 and 3 into equation (11) for $K_{\text{free-free}}$.

(b) Now include K for spring 1 (top fixed) to reach $K_{\text{fixed-free}}$ in (8)

(c) Now place K for spring 4 (bottom fixed) to reach $K_{\text{fixed-fixed}}$ in (7).

9 When $P'(\theta^*) = 0$ for the inverted pendulum, show that $P''(\theta^*) > 0$ which makes θ^* stable. In other words: $\lambda \sin\theta^* = \theta^*$ in (13) gives $\lambda\cos\theta^* < 1$ in (14). For proof, show that $F(\theta) = \theta\cos\theta/\sin\theta$ decreases from $F(0) = 1$ because its derivative is negative. Then $F(\theta^*) = \lambda\cos\theta^* < 1$.

10 The stiffness matrix is $K = A^TCA = D - W$ = diagonal − off-diagonal. It has row sums ≥ 0 and $W \geq 0$. Show that K^{-1} has *positive entries* by checking this identity (the infinite series converges to K^{-1} and all its terms are ≥ 0):

$$KK^{-1} = (D - W)(D^{-1} + D^{-1}WD^{-1} + D^{-1}WD^{-1}WD^{-1} + \cdots) = I.$$

Example 2 All $c_i = c$ and all $m_j = m$ in the fixed-free hanging line of springs. Then

$$A^{\mathrm{T}}CA = c\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \quad \text{and} \quad (A^{\mathrm{T}}CA)^{-1} = \frac{1}{c}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

The displacements $u = K^{-1}f$ change from the fixed-fixed example because K has changed:

Displacements $$u = (A^{\mathrm{T}}CA)^{-1}f = \frac{1}{c}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} mg \\ mg \\ mg \end{bmatrix} = \frac{mg}{c}\begin{bmatrix} \mathbf{3} \\ \mathbf{5} \\ \mathbf{6} \end{bmatrix}.$$

In this fixed-free case, those displacements $3, 5, 6$ are greater than $1.5, 2.0, 1.5$. The number 3 appears in the first displacement u_1 because all three masses are pulling the first spring down. The next mass has an additional displacement $(3 + 2 = 5)$ from the two masses below it. The third mass drops even more $(3 + 2 + 1 = 6)$. The elongations $e = Au$ in the three springs display those numbers 3, 2, 1:

Elongations $$e = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\frac{mg}{c}\begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix} = \frac{mg}{c}\begin{bmatrix} \mathbf{3} \\ \mathbf{2} \\ \mathbf{1} \end{bmatrix}.$$

Multiplying by c, the forces in the three springs are $w_1 = 3mg$ and $w_2 = 2mg$ and $w_3 = mg$. The first spring has three masses below it, the second spring has two, the third spring has one. All springs are now stretched.

The special point of a square matrix A is that those internal forces w can be found directly from the external forces f. The balance equation $A^{\mathrm{T}}w = f$ determines w immediately and uniquely, because $m = n$ and A^{T} is square and invertible:

Spring forces $$w = (A^{\mathrm{T}})^{-1}f \text{ is } \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} mg \\ mg \\ mg \end{bmatrix} = \begin{bmatrix} \mathbf{3}mg \\ \mathbf{2}mg \\ \mathbf{1}mg \end{bmatrix} \begin{matrix} \text{3 masses below} \\ \text{2 masses below} \\ \text{1 mass: free end} \end{matrix}$$

Then e comes from $C^{-1}w$ and u comes from $A^{-1}e$. In this "*determinate*" case $m = n$, we are allowed to write $(A^{\mathrm{T}}CA)^{-1} = A^{-1}C^{-1}(A^{\mathrm{T}})^{-1}$.

Remark 1 When the displacement at the top is fixed by $u_0 = 0$, it requires a force to keep it that way. This is an external *reaction force* f_0, holding up the line of springs. This reaction force is not given in advance. It is part of the output, from force balance at the top. The first spring is pulling down with internal force $w_1 = 3mg$. The reaction $f_0 = -3mg$ pulls upward to balance it. A structural engineer needs to know the reaction forces, to be sure the support will hold and the structure won't collapse.

Example 3 A **FREE-FREE** line of springs has no supports. This means trouble in A and K (spring 1 is gone). The matrix A is 2 by 3, short and wide. Here is $e = Au$:

$$\textbf{Unstable} \quad \begin{bmatrix} e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} u_2 - u_1 \\ u_3 - u_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{9}$$

Now there is a nonzero solution to $Au = 0$. **The masses can move with no stretching of the springs**. The whole line can shift by $u = (1,1,1)$ and this still leaves $e = (0,0)$. The columns of A are *dependent* and the vector $(1,1,1)$ is in the nullspace:

$$\textbf{Rigid motion } \boldsymbol{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad Au = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = e. \tag{10}$$

In this case, $A^{\mathrm{T}}CA$ cannot be invertible. K must be **singular**, because $Au = 0$ certainly leads to $\boldsymbol{A^{\mathrm{T}}CAu = 0}$. The stiffness matrix $A^{\mathrm{T}}CA$ is still square and symmetric, but it is only *positive semidefinite* (like B in Chapter 1, with both ends free):

$$\begin{matrix}\textbf{Singular} \\ \boldsymbol{A^{\mathrm{T}}CA}\end{matrix} \quad \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_2 & \\ & c_3 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix} \tag{11}$$

The pivots will be c_2 and c_3 and *no third pivot*. Two eigenvalues will be positive but the vector $(1,1,1)$ will be an eigenvector for $\lambda = 0$. The matrix is not invertible and we can solve $A^{\mathrm{T}}CAu = f$ only for special vectors f. The external forces have to add to zero, $f_1 + f_2 + f_3 = 0$. Otherwise the whole line of springs (with both ends free) will take off like a rocket.

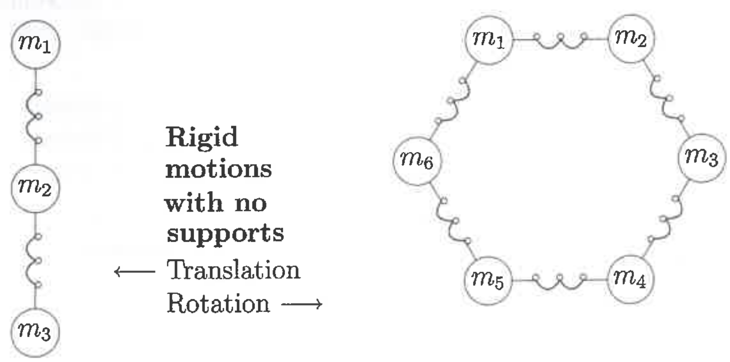

Figure 2.2: The free-free line of springs can move without stretching so $Au = 0$ has nonzero solutions $u = (c,c,c)$. Then $A^{\mathrm{T}}CA$ is singular (also for the "circle" of springs).