1. Enter the matrix $K_5$ by the MATLAB command `toeplitz([2 -1 0 0 0])`.
2. Compute the determinant and the inverse by `det(K)` and `inv(K)`. For a neater answer compute the determinant times the inverse.
3. Find the $L,D,U$ factors of $K_5$ and verify that the $i,j$ entry of $L^{-1}$ is $j/i$.
4. The vector of pivots for $K_4$ is $d = [\frac{2}{3} \frac{3}{4} \frac{4}{5}]$. This is $d = (2:5)/(1:4)$, using MATLAB’s counting vector $i : j = (i,i+1,\ldots,j)$. The extra . makes the division act a component at a time. Find $\ell$ in the MATLAB expression for $L = \text{eye}(4) - \text{diag}(\ell,-1)$ and multiply $L \ast \text{diag}(d) \ast L'$ to recover $K_4$.
5. If $A$ has pivots 2, 7, 6 with no row exchanges, what are the pivots for the upper left 2 by 2 submatrix $B$ (without row 3 and column 3)? Explain why.
6. How many entries can you choose freely in a 5 by 5 symmetric matrix $K$? How many can you choose in a 5 by 5 diagonal matrix $D$ and lower triangular $L$ (with ones on its diagonal)?
7. Suppose $A$ is rectangular ($m$ by $n$) and $C$ is symmetric ($m$ by $m$).
   1. Transpose $A^TCA$ to show its symmetry. What shape is this matrix?
   2. Show why $A^T A$ has no negative numbers on its diagonal.
8. Factor these symmetric matrices into $A = LDL^T$ with the pivots in $D$:

\[
A = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & b \\ b & c \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}
\]

9. The Cholesky command $A = \text{chol}(K)$ produces an upper triangular $A$ with $K = A^T A$. The square roots of the pivots from $D$ are now included on the diagonal of $A$ (so Cholesky fails unless $K = K^T$ and the pivots are positive). Try the chol command on $K_3$, $T_3$, $B_3$, and $B_3 + \text{eps} \ast \text{eye}(3)$.

10. The all-ones matrix $\text{ones}(4)$ is positive semidefinite. Find all its pivots (zero not allowed). Find its determinant and try $\text{eig}(\text{ones}(4))$. Factor it into a 4 by 1 matrix $L$ times a 1 by 4 matrix $L^T$.

11. The matrix $K = \text{ones}(4) + \text{eye}(4)/100$ has all 1’s off the diagonal, and 1.01 down the main diagonal. Is it positive definite? Find the pivots by $\text{lu}(K)$ and eigenvalues by $\text{eig}(K)$. Also find its $LDL^T$ factorization and $\text{inv}(K)$.

12. The matrix $K = \text{pascal}(4)$ contains the numbers from the Pascal triangle (tilted to fit symmetrically into $K$). Multiply its pivots to find its determinant. Factor $K$ into $LL^T$ where the lower triangular $L$ also contains the Pascal triangle!

13. The Fibonacci matrix $[1 \ 1]$ is indefinite. Find its pivots. Factor it into $LDL^T$. Multiply $(1,0)$ by this matrix 5 times, to see the first 6 Fibonacci numbers.
If $A = LU$, solve by hand the equation $Ax = f$ without ever finding $A$ itself. Solve $Lc = f$ and then $Ux = c$ (then $LUx = Lc$ is the desired equation $Ax = f$). $Lc = f$ is forward elimination and $Ux = c$ is back substitution:

$$L = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 2 & 8 & 0 \\ 3 & 5 & 7 \end{bmatrix} \quad f = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}.$$ 

From the multiplication $LS$ show that

$$L = \begin{bmatrix} \ell_{21} & 1 \\ \ell_{31} & 1 \end{bmatrix} \quad S = \begin{bmatrix} 1 & \ell_{21} \\ -\ell_{31} & 1 \end{bmatrix}.$$

$L$ is the inverse of $S$ eliminates multiples of row 1 from lower rows. $L$ adds them back.

Unlike the previous exercise, which eliminated only one column, show that

$$L = \begin{bmatrix} \ell_{21} & 1 \\ \ell_{31} & \ell_{32} \end{bmatrix} \quad S = \begin{bmatrix} 1 & \ell_{21} \\ -\ell_{31} & -\ell_{32} \end{bmatrix}.$$

Write $L$ as $L_1L_2$ to find the correct inverse $L^{-1} = L_2^{-1}L_1^{-1}$ (notice the order):

$$L = \begin{bmatrix} \ell_{21} & 1 \\ \ell_{31} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad L^{-1} = \begin{bmatrix} 1 & -\ell_{21} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\ell_{31} & 1 \end{bmatrix}.$$

By trial and error, find examples of 2 by 2 matrices such that

1. $LU \neq UL$
2. $A^2 = -I,$ with real entries in $A$
3. $B^2 = 0,$ with no zeros in $B$
4. $CD = -DC,$ not allowing $CD = 0$

Write down a 3 by 3 matrix with row 1 - 2 * row 2 + row 3 = 0 and find a similar dependence of the columns—a combination of columns that gives zero.

Draw these equations in their row form (two intersecting lines) and find the solution $(x, y)$. Then draw their column form by adding two vectors:

$$\begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \quad \text{has column form} \quad x \begin{bmatrix} 3 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}.$$

True or false: Every matrix $A$ can be factored into a lower triangular $L$ times an upper triangular $U$, with nonzero diagonals. Find $L$ and $U$ when possible:

When is $A = \begin{bmatrix} 2 & 4 \\ 4 & d \end{bmatrix}$ $LU$? $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $LU$?
Here is one of the most useful formulas in linear algebra (it extends to $T - U V^T$):

\[ \text{Woodbury-Sherman-Morrison} \]

**Inverse of** \( K = T - u v^T \) \hspace{1cm} \[ K^{-1} = T^{-1} + \frac{T^{-1}u v^T T^{-1}}{1 - v^T T^{-1} u} \] (21)

The proof multiplies the right side by \( T - u v^T \), and simplifies to \( I \).

Problem 1.1.7 displays \( T^{-1} - K^{-1} \) when the vectors have length \( n = 4 \):

\[ v^T T^{-1} = \text{row 1 of } T^{-1} = [4 \ 3 \ 2 \ 1] \quad 1 - v^T T^{-1} u = 1 + 4 = 5. \]

For any \( n \), \( K^{-1} \) comes from the simpler \( T^{-1} \) by subtracting \( w^T w/(n+1) \) with \( w = n: -1: 1 \).

**Problem Set 1.4**

1. For \( -u'' = \delta(x - a) \), the solution must be linear on each side of the load. What four conditions determine \( A, B, C, D \) if \( u(0) = 2 \) and \( u(1) = 0 \)?

\[ u(x) = Ax + B \text{ for } 0 \leq x \leq a \text{ and } u(x) = Cx + D \text{ for } a \leq x \leq 1. \]

2. Change Problem 1 to the free-fixed case \( u'(0) = 0 \) and \( u(1) = 4 \). Find and solve the four equations for \( A, B, C, D \).

3. Suppose there are two unit loads, at the points \( a = \frac{1}{3} \) and \( b = \frac{2}{3} \). Solve the fixed-fixed problem in two ways: First combine the two single-load solutions. The other way is to find six conditions for \( A, B, C, D, E, F \):

\[ u(x) = Ax + B \text{ for } x \leq \frac{1}{3}, \quad Cx + D \text{ for } \frac{1}{3} \leq x \leq \frac{2}{3}, \quad Ex + F \text{ for } x \geq \frac{2}{3}. \]

4. Solve the equation \( -d^2 u/dx^2 = \delta(x - a) \) with fixed-free boundary conditions \( u(0) = 0 \) and \( u'(1) = 0 \). Draw the graphs of \( u(x) \) and \( u'(x) \).

5. Show that the same equation with free-free conditions \( u'(0) = 0 \) and \( u'(1) = 0 \) has no solution. The equations for \( C \) and \( D \) cannot be solved. This corresponds to the singular matrix \( B_n \) (with 1,1 and \( n, n \) entries both changed to 1).

6. Show that \( -u'' = \delta(x - a) \) with periodic conditions \( u(0) = u(1) \) and \( u'(0) = u'(1) \) cannot be solved. Again the requirements on \( C \) and \( D \) cannot be met. This corresponds to the singular circulant matrix \( C_n \) (with 1,1 and \( n, n \) entries changed to -1).

7. A difference of point loads, \( f(x) = \delta(x - \frac{1}{3}) - \delta(x - \frac{2}{3}) \), does allow a free-free solution to \( -u'' = f \). Find infinitely many solutions with \( u'(0) = 0 \) and \( u'(1) = 0 \).

8. The difference \( f(x) = \delta(x - \frac{1}{3}) - \delta(x - \frac{2}{3}) \) has zero total load, and \( -u'' = f(x) \) can also be solved with periodic boundary conditions. Find a particular solution \( u_{\text{part}}(x) \) and then the complete solution \( u = u_{\text{part}} + u_{\text{null}} \).
The distributed load \( f(x) = 1 \) is the integral of loads \( \delta(x - a) \) at all points \( x = a \). The free-fixed solution \( u(x) = \frac{1}{2}(1 - x^2) \) from Section 1.3 should then be the integral of the point-load solutions \((1 - x)\) for \( a \leq x \), and \( 1 - a \) for \( a \geq x \):

\[
u(x) = \int_0^x (1-x) \, da + \int_x^1 (1-a) \, da = (1-x)x + (1-\frac{1}{2}x^2) - (x - \frac{1}{2}x^2) = \frac{1}{2} - \frac{1}{2}x^2. \text{YES!}
\]

Check the fixed-fixed case \( u(x) = \int_0^x (1-x) \, da + \int_x^1 (1-a) \, da = \).  

10. If you add together the columns of \( K^{-1} \) (or \( T^{-1} \)), you get a “discrete parabola” that solves the equation \( Ku = f \) (or \( Tu = f \)) with what vector \( f \)? Do this addition for \( K^{-1} \) in Figure 1.9 and \( T^{-1} \) in Figure 1.10.

Problems 11–15 are about delta functions and their integrals and derivatives.

11. The integral of \( \delta(x) \) is the step function \( S(x) \). The integral of \( S(x) \) is the ramp \( R(x) \). Find and graph the next two integrals: the quadratic spline \( Q(x) \) and the cubic spline \( C(x) \). Which derivatives of \( C(x) \) are continuous at \( x = 0 \)?

12. The cubic spline \( C(x) \) solves the fourth-order equation \( u'''' = \delta(x) \). What is the complete solution \( u(x) \) with four arbitrary constants? Choose those constants so that \( u(1) = u''(1) = u(-1) = u''(-1) = 0 \). This gives the bending of a uniform simply supported beam under a point load.

13. The defining property of the delta function \( \delta(x) \) is that

\[
\int_{-\infty}^{\infty} \delta(x) \, g(x) \, dx = g(0) \quad \text{for every smooth function } g(x).
\]

How does this give “area = 1” under \( \delta(x) \)? What is \( \int \delta(x-3) \, g(x) \, dx \)?

14. The function \( \delta(x) \) is a “weak limit” of very high, very thin square waves \( SW \):

\[
SW(x) = \frac{1}{2h} \quad \text{for } |x| \leq h \quad \text{has } \int_{-\infty}^{\infty} SW(x) \, g(x) \, dx \rightarrow g(0) \quad \text{as } h \rightarrow 0.
\]

For a constant \( g(x) = 1 \) and every \( g(x) = x^n \), show that \( \int SW(x) \, g(x) \, dx \rightarrow g(0) \). We use the word “weak” because the rule depends on test functions \( g(x) \).

15. The derivative of \( \delta(x) \) is the doublet \( \delta'(x) \). Integrate by parts to compute

\[
\int_{-\infty}^{\infty} g(x) \delta'(x) \, dx = -\int_{-\infty}^{\infty} (\ ?) \delta(x) \, dx = (\ ?) \text{ for smooth } g(x).
\]
1.5 Eigenvalues and Eigenvectors

Construct $B = B_I$ and $[Q, E] = \text{eig}(B)$ with $B(1, 1) = 1$ and $B(6, 6) = 1$. Verify that $E = \text{diag}(e)$ with eigenvalues $2 \ast \text{ones}(1, 6) - 2 \ast \cos([0 : 5] \ast \text{pi}/6)$ in $e$. How do you adjust $Q$ to produce the (highly important) Discrete Cosine Transform with entries $\text{DCT} = \cos([5 : 5.5] \ast [0 : 5] \ast \text{pi}/6) / \sqrt{3}$?

The free-fixed matrix $T = T_6$ has $T(1, 1) = 1$. Check that its eigenvalues are $2 - 2 \cos ([k - 5/2] \pi/6)$. The matrix $\cos([5 : 5.5] \ast [5 : 5] \ast \text{pi}/6.5) / \sqrt{7.25}$ should contain its unit eigenvectors. Compute $Q' \ast Q$ and $Q' \ast T \ast Q$.

The columns of the Fourier matrix $F_4$ are eigenvectors of the circulant matrix $C = C_4$. But $[Q, E] = \text{eig}(C)$ does not produce $Q = F_4$. What combinations of the columns of $Q$ give the columns of $F_4$? Notice the double eigenvalue in $E$.

Show that the $n$ eigenvalues $2 - 2 \cos \frac{kn}{n+1}$ of $K_n$ add to the trace $2 + \cdots + 2$.

$K_3$ and $B_4$ have the same nonzero eigenvalues because they come from the same $4 \times 3$ backward difference $\Delta_-$. Show that $K_3 = \Delta_-^T \Delta_-$ and $B_4 = \Delta_- \Delta_-^T$. The eigenvalues of $K_3$ are the squared singular values $\sigma^2$ of $\Delta_-$ in 1.7.

Problems 10–23 are about diagonalizing $A$ by its eigenvectors in $S$.

10 Factor these two matrices into $A = SAS^{-1}$. Check that $A^2 = SAS^{-1}$:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}.$$

11 If $A = SAS^{-1}$ then $A^{-1} = \left(\begin{array}{c} s \end{array}\right) \left(\begin{array}{c} s \end{array}\right)^{-1}$. The eigenvectors of $A^3$ are (the same columns of $S$)(different vectors).

12 If $A$ has $\lambda_1 = 2$ with eigenvector $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\lambda_2 = 5$ with $x_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, use $SAS^{-1}$ to find $A$. No other matrix has the same $\lambda$ and $x$.

13 Suppose $A = SAS^{-1}$. What is the eigenvalue matrix for $A + 2I$? What is the eigenvector matrix? Check that $A + 2I = \left(\begin{array}{c} s \end{array}\right) \left(\begin{array}{c} s \end{array}\right)^{-1}$.

14 If the columns of $S$ ($n$ eigenvectors of $A$) are linearly independent, then

(a) $A$ is invertible
(b) $A$ is diagonalizable
(c) $S$ is invertible

15 The matrix $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ is not diagonalizable because the rank of $A - 3I$ is ____. $A$ only has one line of eigenvector. Which entries could you change to make $A$ diagonalizable, with two eigenvectors?

16 $A^k = SAS^{-1}$ approaches the zero matrix as $k \to \infty$ if and only if every $\lambda$ has absolute value less than _____. Which of these matrices has $A^k \to 0$?

$$A_1 = \begin{bmatrix} .6 & .4 \\ .4 & .6 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} .6 & .9 \\ .1 & .6 \end{bmatrix} \quad \text{and} \quad A_3 = K_3.$$
The rabbit and wolf populations show fast growth of rabbits (from 6r) but loss to wolves (from −2w). Find A and its eigenvalues and eigenvectors:

\[
\frac{dr}{dt} = 6r - 2w \quad \text{and} \quad \frac{dw}{dt} = 2r + w.
\]

If \(r(0) = w(0) = 30\) what are the populations at time \(t\)? After a long time, is the ratio of rabbits to wolves 1 to 2 or is it 2 to 1?

Substitute \(y = e^{\lambda t}\) into \(y'' = 6y' - 9y\) to show that \(\lambda = 3\) is a repeated root. This is trouble; we need a second solution after \(e^{3t}\). The matrix equation is

\[
\frac{d}{dt} \begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix}.
\]

Show that this matrix has \(\lambda = 3, 3\) and only one line of eigenvectors. Trouble here too. Show that the second solution is \(y = te^{3t}\).

Explain why \(A\) and \(A^T\) have the same eigenvalues. Show that \(\lambda = 1\) is always an eigenvalue when \(A\) is a Markov matrix, because each row of \(A^T\) adds to 1 and the vector \(\ldots\) is an eigenvector of \(A^T\).

Find the eigenvalues and unit eigenvectors of \(A\) and \(T\), and check the trace:

\[
A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad T = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}.
\]

Here is a quick "proof" that the eigenvalues of all real matrices are real:

\[
Ax = \lambda x \quad \text{gives} \quad x^T Ax = \lambda x^T x \quad \text{so} \quad \lambda = \frac{x^T Ax}{x^T x} \quad \text{is real.}
\]

Find the flaw in this reasoning—a hidden assumption that is not justified.

Find all 2 by 2 matrices that are orthogonal and also symmetric. Which two numbers can be eigenvalues of these matrices?

To find the eigenfunction \(y(x) = \sin k\pi x\), we could put \(y = e^{\lambda x}\) in the differential equation \(-u'' = \lambda u\). Then \(-a^2 e^{ax} = \lambda e^{ax}\) gives \(a = i\sqrt{\lambda}\) or \(a = -i\sqrt{\lambda}\). The complete solution \(y(x) = Ce^{i\sqrt{\lambda} x} + De^{-i\sqrt{\lambda} x}\) has \(C + D = 0\) because \(y(0) = 0\). That simplifies \(y(x)\) to a sine function:

\[
y(x) = C(e^{i\sqrt{\lambda} x} - e^{-i\sqrt{\lambda} x}) = 2iC \sin \sqrt{\lambda} x.
\]

\(y(1) = 0\) yields \(\sin \sqrt{\lambda} = 0\). Then \(\sqrt{\lambda}\) must be a multiple of \(k\pi\), and \(\lambda = k^2 \pi^2\) as before. Repeat these steps for \(y'(0) = y'(1) = 0\) and also \(y'(0) = y(1) = 0\).