Sect 1.3

3
1. Enter the matrix K_5 by the MATLAB command toeplitz([2 −1 0 0 0]).
2. Compute the determinant and the inverse by det(K) and inv(K). For a neater answer compute the determinant times the inverse.
3. Find the L, D, U factors of K_5 and verify that the i, j entry of L^{-1} is j/i.

4 The vector of pivots for K_4 is $d = \left[\frac{2}{1}\ \frac{3}{2}\ \frac{4}{3}\ \frac{5}{4}\right]$. This is $d = (2{:}5)./(1{:}4)$, using MATLAB's counting vector $i : j = (i, i+1, \ldots, j)$. The extra **.** makes the division act *a component at a time*. Find ℓ in the MATLAB expression for $L = \text{eye}(4) - \text{diag}(\ell, -1)$ and multiply $L * \text{diag}(d) * L'$ to recover K_4.

5 If A has pivots 2, 7, 6 with no row exchanges, what are the pivots for the upper left 2 by 2 submatrix B (without row 3 and column 3)? Explain why.

6 How many entries can you choose freely in a 5 by 5 symmetric matrix K? How many can you choose in a 5 by 5 diagonal matrix D and lower triangular L (with ones on its diagonal)?

7 Suppose A is rectangular (m by n) and C is symmetric (m by m).
1. Transpose $A^{\mathrm{T}}CA$ to show its symmetry. What shape is this matrix?
2. Show why $A^{\mathrm{T}}A$ has no negative numbers on its diagonal.

8 Factor these symmetric matrices into $A = LDL^{\mathrm{T}}$ with the pivots in D:

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & b \\ b & c \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

9 The **Cholesky** command $A = \text{chol}(K)$ produces an upper triangular A with $K = A^{\mathrm{T}}A$. The square roots of the pivots from D are now included on the diagonal of A (so Cholesky *fails* unless $K = K^{\mathrm{T}}$ and the pivots are positive). Try the chol command on K_3, T_3, B_3, and $B_3 + \text{eps} * \text{eye}(3)$.

10 The all-ones matrix ones(4) is positive *semidefinite*. Find all its pivots (zero not allowed). Find its determinant and try eig(ones(4)). Factor it into a 4 by 1 matrix L times a 1 by 4 matrix L^{T}.

11 The matrix $K = \text{ones}(4) + \text{eye}(4)/100$ has all 1's off the diagonal, and 1.01 down the main diagonal. Is it positive definite? Find the pivots by lu(K) and eigenvalues by eig(K). Also find its LDL^{T} factorization and inv(K).

12 The matrix K =pascal(4) contains the numbers from the Pascal triangle (tilted to fit symmetrically into K). Multiply its pivots to find its determinant. Factor K into LL^{T} where the lower triangular L also contains the Pascal triangle!

13 The Fibonacci matrix $\left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ is *indefinite*. Find its pivots. Factor it into LDL^{T}. Multiply $(1, 0)$ by this matrix 5 times, to see the first 6 Fibonacci numbers.

2.3

14 If $A = LU$, solve by hand the equation $Ax = f$ without ever finding A itself. Solve $Lc = f$ and then $Ux = c$ (then $LUx = Lc$ is the desired equation $Ax = f$). $Lc = f$ is forward elimination and $Ux = c$ is back substitution:

$$L = \begin{bmatrix} 1 & & \\ 3 & 1 & \\ 0 & 2 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 2 & 8 & 0 \\ & 3 & 5 \\ & & 7 \end{bmatrix} \qquad f = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}.$$

15 From the multiplication LS show that

$$L = \begin{bmatrix} 1 & & \\ \ell_{21} & 1 & \\ \ell_{31} & 0 & 1 \end{bmatrix} \quad \text{is the inverse of} \quad S = \begin{bmatrix} 1 & & \\ -\ell_{21} & 1 & \\ -\ell_{31} & 0 & 1 \end{bmatrix}.$$

S subtracts multiples of row 1 from lower rows. L adds them back.

16 Unlike the previous exercise, which eliminated only one column, show that

$$L = \begin{bmatrix} 1 & & \\ \ell_{21} & 1 & \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix} \quad \text{is } \textit{not} \text{ the inverse of} \quad S = \begin{bmatrix} 1 & & \\ -\ell_{21} & 1 & \\ -\ell_{31} & -\ell_{32} & 1 \end{bmatrix}.$$

Write L as L_1L_2 to find the correct inverse $L^{-1} = L_2^{-1}L_1^{-1}$ (notice the order):

$$L = \begin{bmatrix} 1 & & \\ \ell_{21} & 1 & \\ \ell_{31} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & \ell_{32} & 1 \end{bmatrix} \quad \text{and} \quad L^{-1} = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & -\ell_{32} & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ -\ell_{21} & 1 & \\ -\ell_{31} & 0 & 1 \end{bmatrix}.$$

17 By trial and error, find examples of 2 by 2 matrices such that

1. $LU \neq UL$
2. $A^2 = -I$, with real entries in A
3. $B^2 = 0$, with no zeros in B
4. $CD = -DC$, not allowing $CD = 0$

18 Write down a 3 by 3 matrix with row $1 - 2 *$ row $2 +$ row $3 = 0$ and find a similar dependence of the columns—a combination of columns that gives zero.

19 Draw these equations in their *row* form (two intersecting lines) and find the solution (x, y). Then draw their *column* form by adding two vectors:

$$\begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \quad \text{has column form} \quad x \begin{bmatrix} 3 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}.$$

20 True or false: Every matrix A can be factored into a lower triangular L times an upper triangular U, with *nonzero diagonals*. Find L and U when possible:

$$\text{When is } A = \begin{bmatrix} 2 & 4 \\ 4 & d \end{bmatrix} = LU\,? \qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = LU\,?$$

Here is one of the most useful formulas in linear algebra (it extends to $T - UV^{\mathrm{T}}$):

Woodbury-Sherman-Morrison Inverse of $K = T - uv^{\mathrm{T}}$
$$K^{-1} = T^{-1} + \frac{T^{-1}uv^{\mathrm{T}}T^{-1}}{1 - v^{\mathrm{T}}T^{-1}u} \qquad (21)$$

The proof multiplies the right side by $T - uv^{\mathrm{T}}$, and simplifies to I.

Problem 1.1.7 displays $T^{-1} - K^{-1}$ when the vectors have length $n = 4$:

$$v^{\mathrm{T}}T^{-1} = \text{ row 1 of } T^{-1} = \begin{bmatrix} 4 & 3 & 2 & 1 \end{bmatrix} \qquad 1 - v^{\mathrm{T}}T^{-1}u = 1 + 4 = 5.$$

For any n, K^{-1} comes from the simpler T^{-1} by subtracting $w^{\mathrm{T}}w/(n+1)$ with $w =$n:−1:1.

Problem Set 1.4

1 For $-u'' = \delta(x-a)$, the solution must be linear on each side of the load. What four conditions determine A, B, C, D if $u(0) = 2$ and $u(1) = 0$?

$$u(x) = Ax + B \quad \text{for} \quad 0 \le x \le a \quad \text{and} \quad u(x) = Cx + D \quad \text{for} \quad a \le x \le 1.$$

2 Change Problem 1 to the free-fixed case $u'(0) = 0$ and $u(1) = 4$. Find and solve the four equations for A, B, C, D.

3 Suppose there are *two* unit loads, at the points $a = \frac{1}{3}$ and $b = \frac{2}{3}$. Solve the fixed-fixed problem in two ways: First combine the two single-load solutions. The other way is to find six conditions for A, B, C, D, E, F:

$$u(x) = Ax + B \quad \text{for } x \le \frac{1}{3}, \qquad Cx + D \quad \text{for } \frac{1}{3} \le x \le \frac{2}{3}, \qquad Ex + F \quad \text{for } x \ge \frac{2}{3}.$$

4 Solve the equation $-d^2u/dx^2 = \delta(x-a)$ with **fixed-free** boundary conditions $u(0) = 0$ and $u'(1) = 0$. Draw the graphs of $u(x)$ and $u'(x)$.

5 Show that the same equation with **free-free** conditions $u'(0) = 0$ and $u'(1) = 0$ has no solution. The equations for C and D cannot be solved. This corresponds to the singular matrix B_n (with $1,1$ and n,n entries both changed to 1).

6 Show that $-u'' = \delta(x-a)$ with **periodic** conditions $u(0) = u(1)$ and $u'(0) = u'(1)$ cannot be solved. Again the requirements on C and D cannot be met. This corresponds to the singular circulant matrix C_n (with $1,n$ and $n,1$ entries changed to -1).

7 A *difference* of point loads, $f(x) = \delta(x - \frac{1}{3}) - \delta(x - \frac{2}{3})$, does allow a free-free solution to $-u'' = f$. Find *infinitely many* solutions with $u'(0) = 0$ and $u'(1) = 0$.

8 The difference $f(x) = \delta(x - \frac{1}{3}) - \delta(x - \frac{2}{3})$ has zero total load, and $-u'' = f(x)$ can also be solved with periodic boundary conditions. Find a particular solution $u_{\mathbf{part}}(x)$ and then the complete solution $u_{\mathbf{part}} + u_{\mathbf{null}}$.

9 The distributed load $f(x) = 1$ is the integral of loads $\delta(x-a)$ at all points $x = a$. The free-fixed solution $u(x) = \frac{1}{2}(1-x^2)$ from Section 1.3 should then be the integral of the point-load solutions ($1-x$ for $a \le x$, and $1-a$ for $a \ge x$):

$$u(x) = \int_0^x (1-x)\,da + \int_x^1 (1-a)\,da = (1-x)x + (1-\frac{1^2}{2}) - (x - \frac{x^2}{2}) = \frac{1}{2} - \frac{1}{2}x^2. \text{ YES!}$$

Check the fixed-fixed case $u(x) = \int_0^x (1-x)a\,da + \int_x^1 (1-a)x\,da =$ ____.

10 If you add together the columns of K^{-1} (or T^{-1}), you get a "discrete parabola" that solves the equation $Ku = f$ (or $Tu = f$) with what vector f? Do this addition for K_4^{-1} in Figure 1.9 and T_4^{-1} in Figure 1.10.

Problems 11–15 are about delta functions and their integrals and derivatives.

11 The integral of $\delta(x)$ is the step function $S(x)$. The integral of $S(x)$ is the ramp $R(x)$. Find and graph the next two integrals: the quadratic spline $Q(x)$ and the cubic spline $C(x)$. Which derivatives of $C(x)$ are continuous at $x = 0$?

12 The cubic spline $C(x)$ solves the fourth-order equation $u'''' = \delta(x)$. What is the complete solution $u(x)$ with four arbitrary constants? Choose those constants so that $u(1) = u''(1) = u(-1) = u''(-1) = 0$. This gives the bending of a uniform *simply supported beam* under a point load.

13 The defining property of the delta function $\delta(x)$ is that

$$\int_{-\infty}^{\infty} \delta(x)\, g(x)\, dx = g(0) \quad \text{for every smooth function } g(x).$$

How does this give "area = 1" under $\delta(x)$? What is $\int \delta(x-3)\, g(x)\, dx$?

14 The function $\delta(x)$ is a "weak limit" of very high, very thin square waves SW:

$$SW(x) = \frac{1}{2h} \text{ for } |x| \le h \quad \text{has} \quad \int_{-\infty}^{\infty} SW(x)\, g(x)\, dx \to g(0) \quad \text{as} \quad h \to 0.$$

For a constant $g(x) = 1$ and every $g(x) = x^n$, show that $\int SW(x) g(x)\, dx \to g(0)$. We use the word "weak" because the rule depends on *test functions* $g(x)$.

15 The derivative of $\delta(x)$ is the *doublet* $\delta'(x)$. Integrate by parts to compute

$$\int_{-\infty}^{\infty} g(x)\, \delta'(x)\, dx = -\int_{-\infty}^{\infty} (?)\, \delta(x)\, dx = (??) \text{ for smooth } g(x).$$

5 Construct $B = B_6$ and $[Q, E] = \text{eig}(B)$ with $B(1,1) = 1$ and $B(6,6) = 1$. Verify that $E = \text{diag}(e)$ with eigenvalues $2*\text{ones}(1,6) - 2*\cos([0:5]*\text{pi}/6)$ in e. How do you adjust Q to produce the (highly important) Discrete Cosine Transform with entries $\mathbf{DCT} = \cos([.5:5.5]'*[0:5]*\text{pi}/6)/\text{sqrt}(3)$?

6 The free-fixed matrix $T = T_6$ has $T(1,1) = 1$. Check that its eigenvalues are $2 - 2\cos\left[(k-\frac{1}{2})\pi/6.5\right]$. The matrix $\cos([.5:5.5]'*[.5:5.5]*\text{pi}/6.5)/\text{sqrt}(3.25)$ should contain its unit eigenvectors. Compute $Q'*Q$ and $Q'*T*Q$.

7 The columns of the Fourier matrix F_4 are eigenvectors of the circulant matrix $C = C_4$. But $[Q, E] = \text{eig}(C)$ does not produce $Q = F_4$. What combinations of the columns of Q give the columns of F_4? Notice the double eigenvalue in E.

8 Show that the n eigenvalues $2 - 2\cos\frac{k\pi}{n+1}$ of K_n add to the trace $2 + \cdots + 2$.

9 K_3 and B_4 have the same nonzero eigenvalues because they come from the same 4×3 backward difference Δ_-. Show that $K_3 = \Delta_-{}^{\mathrm{T}}\Delta_-$ and $B_4 = \Delta_-\Delta_-{}^{\mathrm{T}}$. The eigenvalues of K_3 are the squared **singular values** σ^2 of Δ_- in 1.7.

Problems 10–23 are about diagonalizing A by its eigenvectors in S.

10 Factor these two matrices into $A = S\Lambda S^{-1}$. Check that $A^2 = S\Lambda^2 S^{-1}$:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}.$$

11 If $A = S\Lambda S^{-1}$ then $A^{-1} = (\quad)(\quad)(\quad)$. The eigenvectors of A^3 are (the same columns of S)(different vectors).

12 If A has $\lambda_1 = 2$ with eigenvector $x_1 = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $\lambda_2 = 5$ with $x_2 = \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$, use $S\Lambda S^{-1}$ to find A. No other matrix has the same λ's and x's.

13 Suppose $A = S\Lambda S^{-1}$. What is the eigenvalue matrix for $A + 2I$? What is the eigenvector matrix? Check that $A + 2I = (\quad)(\quad)(\quad)^{-1}$.

14 If the columns of S (n eigenvectors of A) are linearly independent, then

(a) A is invertible (b) A is diagonalizable (c) S is invertible

15 The matrix $A = \left[\begin{smallmatrix}3&1\\0&3\end{smallmatrix}\right]$ is not diagonalizable because the rank of $A - 3I$ is ____. A only has one line of eigenvector. Which entries could you change to make A diagonalizable, with two eigenvectors?

16 $A^k = S\Lambda^k S^{-1}$ approaches the zero matrix as $k \to \infty$ if and only if every λ has absolute value less than ____. Which of these matrices has $A^k \to 0$?

$$A_1 = \begin{bmatrix} .6 & .4 \\ .4 & .6 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} .6 & .9 \\ .1 & .6 \end{bmatrix} \quad \text{and} \quad A_3 = K_3.$$

3. 2.3

25 The rabbit and wolf populations show fast growth of rabbits (from $6r$) but loss to wolves (from $-2w$). Find A and its eigenvalues and eigenvectors:

$$\frac{dr}{dt} = 6r - 2w \quad \text{and} \quad \frac{dw}{dt} = 2r + w.$$

If $r(0) = w(0) = 30$ what are the populations at time t? After a long time, is the ratio of rabbits to wolves 1 to 2 or is it 2 to 1?

26 Substitute $y = e^{\lambda t}$ into $y'' = 6y' - 9y$ to show that $\lambda = 3$ is a repeated root. This is trouble; we need a second solution after e^{3t}. The matrix equation is

$$\frac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix}.$$

Show that this matrix has $\lambda = 3, 3$ and only one line of eigenvectors. *Trouble here too.* Show that the second solution is $y = te^{3t}$.

27 Explain why A and A^{T} have the same eigenvalues. Show that $\lambda = 1$ is always an eigenvalue when A is a Markov matrix, because each row of A^{T} adds to 1 and the vector ____ is an eigenvector of A^{T}.

28 Find the eigenvalues and unit eigenvectors of A and T, and check the trace:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad T = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}.$$

29 Here is a quick "proof" that the eigenvalues of all real matrices are real:

$$Ax = \lambda x \quad \text{gives} \quad x^{\mathrm{T}}Ax = \lambda x^{\mathrm{T}}x \quad \text{so} \quad \lambda = \frac{x^{\mathrm{T}}Ax}{x^{\mathrm{T}}x} \quad \text{is real.}$$

Find the flaw in this reasoning—a hidden assumption that is not justified.

30 Find all 2 by 2 matrices that are orthogonal and also symmetric. Which two numbers can be eigenvalues of these matrices?

31 To find the eigenfunction $y(x) = \sin k\pi x$, we could put $y = e^{ax}$ in the differential equation $-u'' = \lambda u$. Then $-a^2e^{ax} = \lambda e^{ax}$ gives $a = i\sqrt{\lambda}$ or $a = -i\sqrt{\lambda}$. The complete solution $y(x) = Ce^{i\sqrt{\lambda}x} + De^{-i\sqrt{\lambda}x}$ has $C + D = 0$ because $y(0) = 0$. That simplifies $y(x)$ to a sine function:

$$y(x) = C(e^{i\sqrt{\lambda}x} - e^{-i\sqrt{\lambda}x}) = 2iC\sin\sqrt{\lambda}x.$$

$y(1) = 0$ yields $\sin\sqrt{\lambda} = 0$. Then $\sqrt{\lambda}$ must be a multiple of $k\pi$, and $\lambda = k^2\pi^2$ as before. *Repeat these steps for $y'(0) = y'(1) = 0$ and also $y'(0) = y(1) = 0$.*