18085 Homework 1

1.1

5 The inverses of K_3 and K_4 (please also invert K_2) have fractions $\frac{1}{\det} = \frac{1}{4}, \frac{1}{5}$:

$$K_3^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} \quad \text{and} \quad K_4^{-1} = \frac{1}{5}\begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 6 & 4 & 2 \\ 2 & 4 & 6 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

First *guess* the determinant of $K = K_5$. Then compute det(K) and inv(K) and det(K)$*$ inv(K)—any software is allowed.

6 (Challenge problem) Find a *formula* for the i, j entry of K_4^{-1} *below the diagonal* ($i \geq j$). Those entries grow linearly along every row and up every column. (Section 1.4 will come back to these important inverses.) Problem 7 below is developed in the Worked Example of Section 1.4.

7 **A column u times a row v^{T} is a rank-one matrix uv^{T}.** All columns are multiples of u, and all rows are multiples of v^{T}. $T_4^{-1} - K_4^{-1}$ has rank 1:

$$T_4^{-1} - K_4^{-1} = \frac{1}{5}\begin{bmatrix} 16 & 12 & 8 & 4 \\ 12 & 9 & 6 & 3 \\ 8 & 6 & 4 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}\begin{bmatrix} 4 & 3 & 2 & 1 \end{bmatrix}$$

Write $K_3 - T_3$ in this special form uv^{T}. Predict a similar formula for $T_3^{-1} - K_3^{-1}$.

8 (a) Based on Problem 7, predict the i, j entry of $T_5^{-1} - K_5^{-1}$ below the diagonal.

(b) Subtract this from your answer to Problem 1 (the formula for T_5^{-1} when $i \geq j$). This gives the not-so-simple formula for K_5^{-1}.

9 Following Example **1.1 A** with C instead of B, show that $e = (1, 1, 1, 1)$ is perpendicular to each column of C_4. Solve $Cu = f = (1, -1, 1, -1)$ with the singular matrix C by u = pinv(C) $*$ f. Try u = C\e and C\f, before and after adding a fifth equation $0 = 0$.

10 The "hanging matrix" H in Worked Example **1.1 B** changes the last entry of K_3 to $H_{33} = 1$. Find the inverse matrix from $H^{-1} = JT^{-1}J$. Find the inverse also from $H = UU^{\mathrm{T}}$ (check upper times lower triangular!) and $H^{-1} = (U^{-1})^{\mathrm{T}}U^{-1}$.

11 Suppose U is any upper triangular matrix and J is the reverse identity matrix in **1.1 B**. Then JU is a "southeast matrix". What geographies are UJ and JUJ? By experiment, a southeast matrix times a northwest matrix is ____.

12 Carry out elimination on the 4 by 4 circulant matrix C_4 to reach an upper triangular U (or try $[L, U] = \mathrm{lu}(C)$ in MATLAB). Two points to notice: The last entry of U is ____ because C is singular. The last column of U has new nonzeros. Explain why this "fill-in" happens.

13 By hand, can you factor the circulant C_4 (with three nonzero diagonals, allowing wraparound) into circulants L times U (with two nonzero diagonals, allowing wraparound so not truly triangular)?

14 Gradually reduce the diagonal $2, 2, 2$ in the matrix K_3 until you reach a singular matrix M. This happens when the diagonal entries reach ____. Check the determinant as you go, and find a nonzero vector that solves $Mu = 0$.

Questions 15–21 bring out important facts about matrix multiplication.

15 How many individual multiplications to create Ax and A^2 and AB?

$$A_{n \times n}\, x_{n \times 1} \qquad A_{n \times n}\, A_{n \times n} \qquad A_{m \times n}\, B_{n \times p} = (AB)_{m \times p}$$

16 You can multiply Ax by rows (the usual way) or **by columns** (more important). Do this multiplication both ways:

$$\textbf{By rows} \qquad \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \text{inner product using row 1} \\ \text{inner product using row 2} \end{bmatrix}$$

$$\textbf{By columns} \qquad \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 2 \\ 4 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} \textbf{combination} \\ \textbf{of columns} \end{bmatrix}$$

17 The product Ax is a **linear combination of the columns of** A. The equations $Ax = b$ have a solution vector x exactly when b is a ____ of the columns.

Give an example in which b is *not in the column space* of A. There is no solution to $Ax = b$, because b is not a combination of the columns of A.

18 Compute $C = AB$ by multiplying the matrix A times each column of B:

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 8 & * \\ 14 & * \end{bmatrix}.$$

Thus, A * B(:,j) = C(:,j).

19 You can also compute AB by multiplying each row of A times B:

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 * \text{row } 1 + 3 * \text{row } 2 \\ 4 * \text{row } 1 + 5 * \text{row } 2 \end{bmatrix} = \begin{bmatrix} 8 & 16 \\ * & * \end{bmatrix}.$$

A solution to $Bx = 0$ is also a solution to $(AB)x = 0$. Why? From

$$Bx = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{how do we know} \quad ABx = \begin{bmatrix} 8 & 16 \\ * & * \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}?$$

20 The four ways to find AB give numbers, columns, rows, and **matrices**:

1 (*rows of* A) times (*columns of* B)	C(i,j) = A(i,:) * B(:,j)
2 A times (*columns of* B)	C(:,j) = A * B(:,j)
3 (*rows of* A) times B	C(i,:) = A(i,:) * B
4 (columns of A) times (rows of B)	for k = 1:n, C = C + A(:,k) * B(k,:); end

Finish these 8 multiplications for **columns times rows**. How many for n by n?

$$\begin{bmatrix}2&3\\4&5\end{bmatrix}\begin{bmatrix}1&2\\2&4\end{bmatrix}=\begin{bmatrix}2\\4\end{bmatrix}\begin{bmatrix}1&2\end{bmatrix}+\begin{bmatrix}3\\5\end{bmatrix}\begin{bmatrix}2&4\end{bmatrix}=\begin{bmatrix}2&4\\4&8\end{bmatrix}+\begin{bmatrix}*&*\\*&*\end{bmatrix}=\begin{bmatrix}8&*\\*&*\end{bmatrix}.$$

21 Which *one* of these equations is true for all n by n matrices A and B?

$$AB = BA \qquad (AB)A = A(BA) \qquad (AB)B = B(BA) \qquad (AB)^2 = A^2B^2.$$

22 Use $n = 1000$; $e =$ ones$(n, 1)$; $K =$ spdiags$([-e, 2*e, -e], -1:1, n, n)$; to enter K_{1000} as a sparse matrix. Solve the sparse equation $Ku = e$ by u = K\e. Plot the solution by plot(u).

23 Create 4-component vectors u, v, w and enter $A =$ spdiags$([u, v, w], -1:1, 4, 4)$. Which components of u and w are left out from the -1 and 1 diagonals of A?

24 Build the sparse identity matrix $I =$ sparse$(i, j, s, 100, 100)$ by creating vectors i, j, s of positions i, j with nonzero entries s. (You could use a for loop.) In this case speye(100) is quicker. Notice that sparse(eye(10000)) would be a disaster, since there isn't room to store eye(10000) before making it sparse.

25 The only solution to $Ku = 0$ or $Tu = 0$ is $u = 0$, so K and T are invertible. For proof, suppose u_i is the largest component of u. If $-u_{i-1} + 2u_i - u_{i+1}$ is zero, this forces $u_{i-1} = u_i = u_{i+1}$. Then the next equations force every $u_j = u_i$. At the end, when the boundary is reached, $-u_{n-1} + 2u_n$ only gives zero if $u = 0$.

Why does this "diagonally dominant" argument fail for B and C?

26 For which vectors v is toeplitz(v) a circulant matrix (cyclic diagonals)?

27 (Important) Show that the 3 by 3 matrix K comes from $A_0^{\mathrm{T}}A_0$:

$$A_0 = \begin{bmatrix}-1&1&0&0\\0&-1&1&0\\0&0&-1&1\end{bmatrix} \quad \text{is a } \textbf{“difference matrix”}$$

Which column of A_0 would you remove to produce A_1 with $T = A_1^{\mathrm{T}}A_1$? Which column would you remove next to produce A_2 with $B = A_2^{\mathrm{T}}A_2$? The difference matrices A_0, A_1, A_2 have $0, 1, 2$ boundary conditions. So do the "second differences" K, T, and B.

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Problem Set 1.2

1 What are the second derivative $u''(x)$ and the second difference $\Delta^2 U_n$? Use $\delta(x)$.

$$u(x) = \begin{cases} Ax & \text{if } x \leq 0 \\ Bx & \text{if } x \geq 0 \end{cases} \qquad U_n = \begin{cases} An & \text{if } n \leq 0 \\ Bn & \text{if } n \geq 0 \end{cases} = \begin{bmatrix} -2A \\ -A \\ 0 \\ B \\ 2B \end{bmatrix}$$

$u(x)$ and U are piecewise linear with a corner at 0.

2 Solve the differential equation $-u''(x) = \delta(x)$ with $u(-2) = 0$ and $u(3) = 0$. The pieces $u = A(x+2)$ and $u = B(x-3)$ meet at $x = 0$. Show that the vector $U = (u(-1), u(0), u(1), u(2))$ solves the corresponding matrix problem $KU = F = (0, 1, 0, 0)$.

Problems 3–12 are about the "local accuracy" of finite differences.

3 The h^2 term in the error for a centered difference $(u(x+h) - u(x-h))/2h$ is $\frac{1}{6}h^2 u'''(x)$. Test by computing that difference for $u(x) = x^3$ and x^4.

4 Verify that the inverse of the backward difference matrix Δ_- in (28) is the sum matrix in (29). But the centered difference matrix $\Delta_0 = (\Delta_+ + \Delta_-)/2$ might not be invertible! Solve $\Delta_0 u = 0$ for $n = 3$ and $n = 5$.

5 In the Taylor series (2), find the number a in the next term $ah^4 u''''(x)$ by testing $u(x) = x^4$ at $x = 0$.

6 For $u(x) = x^4$, compute the second derivative and second difference $\Delta^2 u/(\Delta x)^2$. From the answers, predict c in the leading error in equation (9).

7 Four samples of u can give fourth-order accuracy for du/dx at the center:

$$\frac{-u_2 + 8u_1 - 8u_{-1} + u_{-2}}{12h} = \frac{du}{dx} + bh^4 \frac{d^5 u}{dx^5} + \cdots$$

1. Check that this is correct for $u = 1$ and $u = x^2$ and $u = x^4$.
2. Expand u_2, u_1, u_{-1}, u_{-2} as in equation (2). Combine the four Taylor series to discover the coefficient b in the h^4 leading error term.

8 *Question* Why didn't I square the centered difference for a good Δ^2?

Answer A centered difference of a centered difference stretches too far:

$$\frac{\Delta_0}{2h}\frac{\Delta_0}{2h} u_n = \frac{u_{n+2} - 2u_n + u_{n-2}}{(2h)^2}$$

The second difference matrix now has $\mathbf{1, 0, -2, 0, 1}$ on a typical row. The accuracy is no better and we have trouble with u_{n+2} at the boundaries.

Can you construct a fourth-order accurate centered difference for d^2u/dx^2, choosing the right coefficients to multiply $u_2, u_1, u_0, u_{-1}, u_{-2}$?