18.085 PSet 5 Solutions.

1. (1)

$$\text{curl } \vec{F_1} = \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) - \frac{\partial}{\partial x}\left(\frac{-x}{x^2+y^2}\right)$$

$$= \frac{1}{x^2+y^2} - \frac{y \cdot 2y}{(x^2+y^2)^2} + \frac{1}{x^2+y^2} - \frac{x \cdot 2x}{(x^2+y^2)^2}$$

$$= \frac{2}{x^2+y^2} - \frac{2(x^2+y^2)}{(x^2+y^2)^2}$$

$$= \boxed{0.}$$

(2) We parameterize the unit circle by

$$\begin{cases} x = \cos t \\ y = \sin t \end{cases} \quad 0 \le t \le 2\pi$$

So the work

$$W_1 = \int_{\text{unit circle}} \frac{y}{x^2+y^2}\,dx - \frac{x}{x^2+y^2}\,dy$$

$$= \int_0^{2\pi} \sin t \, d\cos t - \cos t \, d\sin t$$

$$= -\int_0^{2\pi} (\sin^2 t + \cos^2 t)\, dt$$

$$\boxed{= -2\pi}$$

(3). As the work done by $\vec{F_1}$ around a closed loop (unit circle) is not 0, we know that $\vec{F_1}$ is <u>not</u> a gradient vector field (conservative force field).

(4). $$\text{curl } \vec{F_2} = \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right)$$

$$= -\frac{2xy}{(x^2+y^2)^2} + \frac{2xy}{(x^2+y^2)^2} = \boxed{0}.$$

So $\vec{F_2}$ is irrotational.

The work done is

$$W_2 = \int_{\text{unit circle}} \frac{x}{x^2+y^2}\,dx + \frac{y}{x^2+y^2}\,dy$$

$$= \int_0^{2\pi} \cos t \, d\cos t + \sin t \, d\sin t$$

$$= \int_0^{2\pi} 0 \cdot dt = \boxed{0}.$$

$\vec{F_2}$ is a gradient, a potential function is

$$u(x,y) = \frac{1}{2} \ln (x^2+y^2)$$

check $\vec{F_2} = \text{grad } u$.

2. (1) The matrix is

$$A = \begin{pmatrix} 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{pmatrix}$$

is anti-symmetric,

i.e. $A^T = -A$.

Any 3×3 anti-symmetric matrix is singular,

since $A^T = -A$

taking det,

we get

$$\begin{array}{ccc} \det A^T & = & \det(-A) \\ \| & & \| \; (3\times3) \\ \det A & = & -\det A \end{array}$$

$\Rightarrow \det A = 0.$

Notice that $\operatorname{curl}(0, -xz, 0) = (x, 0, -z)$

$\operatorname{curl}(yz, 0, 0) = (0, y, -z)$

So a vector potential for $(2x, 3y, -5z)$

$$= 2(x, 0, -z) + 3(0, y, -z)$$

is $2(0, -xz, 0) + 3(yz, 0, 0) = (3yz, -2xz, 0)$

We can take $\boxed{S_1 = (3yz, -2xz, 0)}$

and $\boxed{S_2 = S_1 + \text{grad } f}$

for any function f.

(2) By definition

$$\widehat{\nabla f} = \widehat{\text{grad} f} = \widehat{(f_x, f_y, f_z)}$$

$$= (\widehat{f_x}, \widehat{f_y}, \widehat{f_z}) = (i\xi_x \cdot \hat{f}, i\xi_y \cdot \hat{f}, i\xi_z \cdot \hat{f})$$

$= \cancel{(i \widehat{f_x}\, f_y}$

$$= i\,(\xi_x, \xi_y, \xi_z)\cdot \hat{f}$$

$$= \boxed{i\,\xi \cdot \hat{f}}$$

Similarly, $\widehat{\nabla \times g} = \widehat{\operatorname{curl} g}$

$$= \left(\widehat{\frac{\partial g_3}{\partial y} - \frac{\partial g_2}{\partial z}},\ \widehat{\frac{\partial g_1}{\partial z} - \frac{\partial g_3}{\partial x}},\ \widehat{\frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y}} \right)$$

$$= \left(i\xi_y\cdot\hat{g}_3 - i\xi_z\cdot\hat{g}_2,\ i\xi_z\cdot\hat{g}_1 - i\xi_x\hat{g}_3,\ i\xi_x\hat{g}_2 - i\xi_y\hat{g}_1 \right)$$

$$= i\,(\xi_x, \xi_y, \xi_z) \times (\hat{g}_1, \hat{g}_2, \hat{g}_3)$$

$$= \boxed{i\,\xi \times \hat{g}}$$

(3). If g is in the null space of curl

i.e. $\text{curl}\, g = 0$

$$\Updownarrow$$

$$0 = \widehat{\text{curl}\, g} = i\xi \times \hat{g} \qquad \text{by (2)}.$$

Notice that $\xi \times \hat{g} = 0$ if and only if $\xi \parallel \hat{g}$.
(property of cross product)

i.e. $\hat{g}$ can be written as.

$$\hat{g} = i\xi \cdot \hat{f} \quad \text{for some function}$$

Using (2) again (and Fourier transform is invertible)

$$\hat{g} = i\xi \cdot \hat{f} = \widehat{\text{grad} f}$$

$\Rightarrow g = \text{grad} f$. | So any vector in the null space is a gradient! |

Problem 3 Soln:

Define

$$\mathcal{H}(z,\omega) = G(z,\omega) - G(z,\bar{\omega})$$

where

$$G(z,\omega) = \frac{-1}{2\pi} \log|z-\omega|$$

then

$$\Delta \mathcal{H}(z,\omega) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\mathcal{H} = \Delta G(z,\omega) - \Delta G(z,\bar{\omega})$$

$$= \delta(z-\omega) - \delta(z-\bar{\omega})$$

if z, ω have imaginary part > 0,

then $\delta(z-\bar{\omega}) = 0$ since $\bar{\omega}$ has imaginary part < 0.

So $\Delta \mathcal{H}(z,\omega) = \delta(z-\omega)$

next

$$\mathcal{H}(x,\omega) = G(x,\omega) - G(x,\bar{\omega}) = 0$$

where x is just a real number.

Thus $\mathcal{H}$ is our Green function.

Problem 4 Soln:

Let $\gamma(t) = (\underbrace{u + r\cos t}_{x(t)}, \underbrace{v + r\sin t}_{y(t)})$, $0 \le t \le 2\pi$

now

$$\frac{1}{2\pi r}\int_{C_{r,(u,v)}} f(\gamma)\, d\gamma = \frac{1}{2\pi r}\int_0^{2\pi} (x^2(t) - y^2(t)) \underbrace{\sqrt{\dot{x}^2(t) + \dot{y}^2(t)}}_{\sqrt{r^2\sin^2 t + r^2\cos^2 t} = r}\, dt$$

$$= \frac{1}{2\pi \cancel{r}}\int_0^{2\pi} \left((u + r\cos t)^2 - (v + r\sin t)^2\right)\cancel{r}\, dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \left(u^2 + 2r\cos t\, u + r^2\cos^2 t - v^2 - \overset{2rv\sin t}{\cancel{2rv\sin t}} - r^2\sin^2 t\right) dt$$

$$= u^2 - v^2 + \frac{1}{2\pi}\int_0^{2\pi} (u \cdot 2r\cos t - v\, 2r\sin t)\, dt$$

$$+ \frac{1}{2\pi}\int_0^{2\pi} r^2(\cos^2 t - \sin^2 t)\, dt$$

We see $\int_0^{2\pi} \cos t\, dt = \int_0^{2\pi} \sin t\, dt = 0$

using) $\cos^2 t = \frac{1 + \cos 2t}{2}$ $\quad \sin^2 t = \frac{1 - \cos 2t}{2}$

$$\Rightarrow \int_0^{2\pi} \cos^2 t - \sin^2 t\, dt = \int_0^{2\pi} \cos 2t\, dt = 0.$$

Problem 5

(1) For a matrix Π of size $n^2 \times n^2$, we use $[\Pi]_{ij}$ to denote the ij-th block of size $n \times n$.

Let A, B, C, D be $n \times n$ matrices

$$
\begin{aligned}
\left[k(A,B)\cdot k(C,D)\right]_{ij} &= \sum_{k=1}^{n} \left[k(A,B)\right]_{ik} \cdot \left[k(C,D)\right]_{kj} \\
&= \sum_{k=1}^{n} a_{ik} B \cdot c_{kj} D \\
&= \left(\sum_{k=1}^{n} a_{ik} c_{kj}\right) BD \\
&= (AC)_{ij}\, BD \\
&= \left[k(AC, BD)\right]_{ij}
\end{aligned}
$$

$$\boxed{k(A,B)\cdot k(C,D) = k(AC, BD)}$$

If A, B are invertible, $k(A,B)\cdot k(A^{-1}, B^{-1}) = k(AA^{-1}, BB^{-1}) = k(I_n, I_n) = I_{n^2}$

$$\boxed{(k(A,B))^{-1} = k(A^{-1}, B^{-1})}$$

$$\left[k(A,B)^T\right]_{ij} = \left[k(A,B)\right]_{ji}^T = a_{ji} B^T = (A^T)_{ij}\, B^T = \left[k(A^T, B^T)\right]_{ij}$$

$$\boxed{k(A,B)^T = k(A^T, B^T)}$$

(2) for any node in the grid, $\left(h = \frac{1}{n+1}\right)$

$$4U_{ij} - U_{i+1,j} - U_{i-1,j} - U_{i,j+1} - U_{i,j-1} = 4h^2$$

Define: $F_{ij} = 4h^2$ if $1 < i,j < n$

$F_{11} = 4h^2 + U_{01} + U_{10}$

$F_{1n} = 4h^2 + U_{0n} + U_{1,n+1}$

$F_{n1} = 4h^2 + U_{n0} + U_{n+1,1}$

$F_{nn} = 4h^2 + U_{n,n+1} + U_{n+1,n}$

$F_{1i} = 4h^2 + U_{0i} \quad 1 < i < n$

$F_{ni} = 4h^2 + U_{n+1,i} \quad$ "

$F_{i1} = 4h^2 + U_{i0} \quad$ "

$F_{in} = 4h^2 + U_{i,n+1} \quad$ "

$$F = \begin{pmatrix} F_{11} \\ \vdots \\ F_{1n} \\ F_{21} \\ \vdots \\ F_{2n} \\ \vdots \\ F_{n1} \\ \vdots \\ F_{nn} \end{pmatrix}$$

idem for U

$(K2D)\,U = F$. ~~K2D is a symmetr~~

eigenvalues of $(K2D)$: $\lambda_{kl} = 4 - 2\left(\cos\frac{k\pi}{n+1} + \cos\frac{l\pi}{n+1}\right)$

corresponding eigenvector. $(y_{kl})_{ij} = \sin\frac{ik\pi}{n+1} \cdot \sin\frac{jl\pi}{n+1}$

$(K2D) = S\Lambda S^{-1}$.

Step 1: Compute $S^{-1}F$

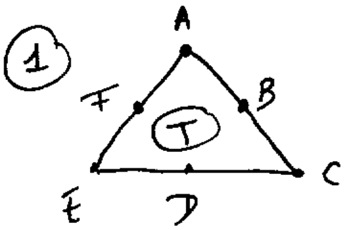
Step 2: multiply by Λ^{-1}

Step 3: multiply by S

(3) $f(x) = 1 - x^2 - y^2$ satisfies $\Delta f = -4$ and the boundary conditions. $U_{kk} \xrightarrow[k \to \infty]{} f\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}$

Problem 6.

(1)

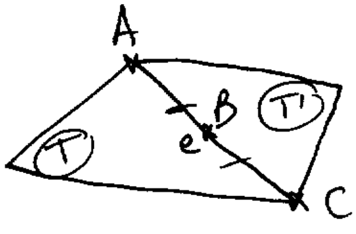


We are given parameters $\alpha_A, \alpha_B, \dots, \alpha_F$ and we want to find a second order polynomial ϕ_T such that $\phi_T(A) = \alpha_A, \dots, \phi_T(F) = \alpha_F$.

For any point $P \in \{A, \dots, F\}$ we can find two lines D_P^1 and D_P^2 which do not contain P but contain all other points. Let us define $\phi^P = D_P^1 \cdot D_P^2$ (recall that the cartesian equation of a line is a degree one polynomial in x and y, so ϕ^P is of degree 2).

Then $\phi^P(P) \neq 0$ and $\phi^P(Q) = 0$ for $Q \neq P$.

$$\phi_T^* = \frac{\alpha_A}{\phi^A(A)} \phi^A + \dots + \frac{\alpha_F}{\phi^F(F)} \phi^F \text{ will work.}$$



Restricted on e, ϕ_T and $\phi_{T'}$ are second order polynomials in one variable (the parameter of the edge) which agree at A, B, C. Hence they agree on the whole edge e.

(2) check that the code works for $\Delta u = -1$ and change the grid parameters to 5, 5.