Pset 4 18.085

Solutions

Problem 1:

A is an incidence matrix implies row j ~~each row~~ has a -1 at the vertex (labled from $1, \dots, n$) where the edge begins and a $+1$ where the edge terminates (with 0's in every other component). This implies each row sum is 0, therefore $A\vec{c1} = 0$ ~~$\forall c \in \mathbb{R}$~~. for any constant c. As stated in the section, the rank of A when A is the incidence matrix of a complete graph is $n-1$, this means the dimension of the null space is 1, i.e. $\vec{c1}$ is the only solution to $A\vec{c1} = 0$.

Problem 2:

Since $A^T A = D - W$, where D is the degree matrix and W is the adjacency matrix

$$Tr(A^T A) = Tr(D) - Tr(W)$$

$$Tr(D) = \sum_{i=1}^{n} \left(\cancel{\text{degrees}} \text{ Number of edges that meet node } i \right)$$

$$= 2m$$

twice the total # of edges

because we count each edge twice (once for the node it starts from and once for the node it ends at)

$Tr(W) = 0$ because ~~we do not allow edges to connect to themselves.~~

W has no diagonal elements.

18.085 Problem Set 4. Solutions

Problem 3

(1) A is a $\boxed{8\times 8 \text{ matrix}}$ since the truss has 8 bars and 4 free nodes.

We can write down the matrix A explicitly:
(assuming all the angles are multiples of 45°)

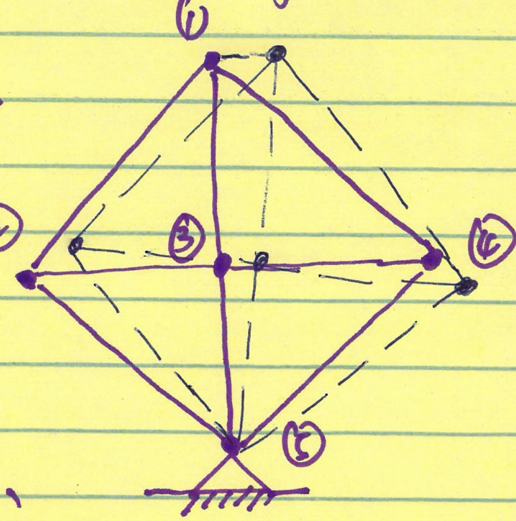
$$A = \begin{pmatrix} \frac{1}{\sqrt2} & \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} & -\frac{1}{\sqrt2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ -\frac{1}{\sqrt2} & \frac{1}{\sqrt2} & 0 & 0 & 0 & 0 & \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{\sqrt2} & \frac{1}{\sqrt2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \end{pmatrix}$$

We can read off the 1^{st} column.

The truss has a rigid motion of rotating around node 5.

As A is a square matrix, and truss has a rigid motion we know: $\det A = 0$

$\Rightarrow \det(A^T) = 0 \Rightarrow A^T w$ has a nonzero solution.

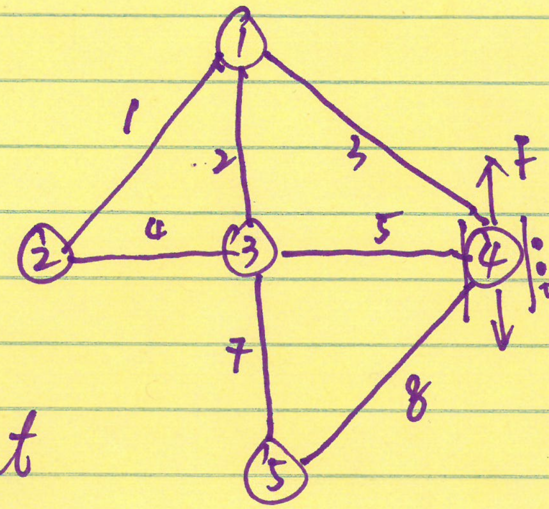


(2).

$$A' = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & -1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 & 0 & 0 & -1/\sqrt{2} \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/\sqrt{2} \end{pmatrix}$$

It is easy to check that $\det A' \neq 0$.

The truss represented by A' is obtained by removing bar 6 from the original truss and restricting node 4 only moving in the vertical direction.



As A' is non-singular,

the corresponding truss is stable

so the columns of A' are independent

$\Rightarrow$ columns 1, 2, 3, 4, 5, 6, 8 of A are independent

$\Rightarrow \operatorname{rk}(A) \geq 7$.

Notice $\operatorname{rk}(A) \neq 8$ (otherwise A is nonsingular)

we conclude $\boxed{\operatorname{rank}(A) = 7}$

(3) As $\operatorname{rank}(A) = 7$, its null space is of dimension $8 - \operatorname{rank}(A) = 1$, consisting of the rigid motion described in (1).

So truss D in (1) does not have any mechanism.

Problem 4

(1) $\dot{A}$ has size $\boxed{m \times 3n_2}$,

where each row represents a bar,

each 3 columns represent a free node

since every free node has 3 components of coordinates.

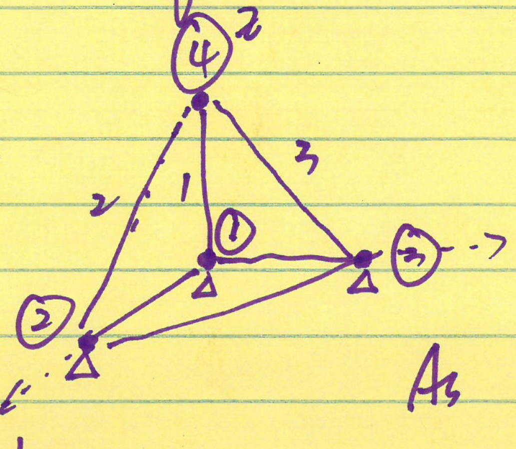
(2). 3D rigid motions ~~has~~ are of dimension $\boxed{6}$.

they are generated by translation along x, y, z directions

and rotating around x, y, z - axes.

(3) There are at least $\boxed{3}$ fixed nodes.

If there are only 2 fixed nodes, then rotating around the axis passing through these 2 nodes is a rigid motion.

(4). 

The truss is shown in the left figure.

As nodes ①, ②, ③ are fixed, we can ignore the bars connecting them. since the corresponding row in A is 0.

$$A = \begin{pmatrix} 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

has independent columns.

$\boxed{\text{so the truss is stable}}$

Problem 5

(1) $u(x) = \frac{1}{6}(x - x^3)$.

weak form: $\int_0^1 u'v' = \int_0^1 xv$

Let ϕ_1 and ϕ_2 be the hat functions centered at $h = \frac{1}{3}$ and $h = \frac{2}{3}$.

$K = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix}$ $F = \begin{pmatrix} 1/9 \\ 2/9 \end{pmatrix}$

$K^{-1} = \frac{1}{9}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. $K^{-1}F = \frac{1}{81}\begin{pmatrix} 4 \\ 5 \end{pmatrix}$

The approximation U of u is $\frac{4}{81}\phi_1 + \frac{5}{81}\phi_2$.

$u\left(\frac{1}{3}\right) = \frac{1}{6}\left(\frac{1}{3} - \frac{1}{27}\right) = \frac{1}{6} \cdot \frac{8}{27} = \frac{4}{81}$

$u\left(\frac{2}{3}\right) = \frac{1}{6}\left(\frac{2}{3} - \frac{8}{27}\right) = \frac{1}{6} \cdot \frac{10}{27} = \frac{5}{81}$

$\Rightarrow$ U and u agree on the nodes.

cartesian equations for $u - U$:

$\left[0, \frac{1}{3}\right]$: $\frac{1}{6}(x - x^3) - \frac{4}{27}x$

$\left[\frac{1}{3}, \frac{2}{3}\right]$: $\frac{1}{6}(x - x^3) - \frac{1}{27}(x+1)$

$\left[\frac{2}{3}, 1\right]$: $\frac{1}{6}(x - x^3) - \frac{5}{27}(1 - x)$

The maximum is attained at $x^* = \frac{1}{3}\sqrt{\frac{19}{3}}$ $(U - u)(x^*) = \frac{19}{243}\sqrt{\frac{19}{3}} - \frac{5}{27} \simeq 0{,}01$

(2) equation for ϕ: $\phi(x) = \frac{4(x - 3h)(4h - x)}{h^2}$. ϕ and ϕ'^2 are quadratic hence Simpson's rule holds:

$$\int_{3h}^{4h} \phi = \frac{h}{6}(0 + 4 + 0) = \frac{2h}{3}$$

$$\int_{3h}^{4h} \phi'^2 = \frac{h}{6}\left(\phi'(3h)^2 + 0 + \phi'(4h)^2\right) = \frac{16}{3h}$$

③ let $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ where x_i are real numbers.

$$X^T K X = \int_0^1 \left(\sum x_i \phi_i'\right)^2 \geq 0$$

$X^T K X = 0$ implies $\sum x_i \phi_i'$ identically 0 on $[0,1]$.
The function $\sum x_i \phi_i$ is constant. Its value at 0 is 0.
Hence $\sum x_i \phi_i$ is identically 0 on $[0,1]$. Since the ϕ_i's are linearly independent, all the x_i's must be 0. $\boxed{X = 0}$.

$X^T K X > 0$ for all $X \neq 0$. K is positive definite.

Problem 6

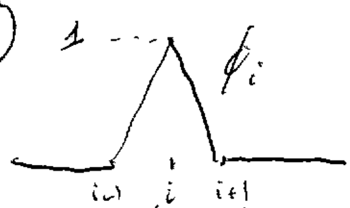
① $u(x) = (\alpha x + \beta) e^{-x} + x \quad u(0) = u(1) = 0$ hence $u(x) = x(1 - e^{1-x})$

weak formulation of (P): $\int_0^1 (u'v' + 2uv' - uv) = -\int_0^1 (x+2) v$ for $v(0) = v(1) = 0$

$u_\Phi = \sum U_i \phi_i$ is defined to be the unique solution to the preceding equation for $v = \phi_1, \dots, \phi_N$. We have $KU = F$ with $F_i = -\int_0^1 (x+2)\phi_i$ and

$$K_{ij} = \int_0^1 \phi_i' \phi_j' + 2\int_0^1 \phi_j \phi_i' - \int_0^1 \phi_i \phi_j$$

②



if $f(x) = \alpha_1 \phi_1(x) + \dots + \alpha_{n-1} \phi_{n-1}(x) = 0$
$f(\frac{i}{n}) = \alpha_i = 0$. Therefore the ϕ_i's are independent.

$K_{ii} = 2n - \frac{4}{3n}$ for $i = 1 \dots n-1$

$K_{i,i+1} = -n + 1 - \frac{1}{6n}$ for $i = 1 \dots n-2$

$K_{i+1,i} = -n + 1 - \frac{1}{6n}$ for $i = 1 \dots n-2$

all the other coefficients are 0.

$$F_i = \int \phi_i\,(x+2) = \int_{\frac{i-1}{n}}^{\frac{i+1}{n}} \phi_i(x)\,(x+2) = \frac{1}{3n}\cancel{\left(4\left(\frac{i}{n}+2\right)\right)}$$

$$= \cancel{\frac{8}{3n} + \frac{4i}{3n^2}} = \cancel{\frac{8n+4i}{3n^2}}$$

$$= \frac{2n+i}{n^2}$$

(use Simpson's rule on $\left[\frac{i-1}{n}, \frac{i}{n}\right]$ and $\left[\frac{i}{n}, \frac{i+1}{n}\right]$)

3

$$K = \begin{pmatrix} \frac{106}{9} & -7-\frac{1}{36} & 0 & 0 & 0 \\ -5+\frac{1}{36} & \ddots & \ddots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & \ddots & \frac{106}{9} \end{pmatrix}$$

$$F = \begin{pmatrix} 13/36 \\ 14/36 \\ 15/36 \\ 16/36 \\ 17/36 \end{pmatrix} \qquad U = K^{-1}F$$