Problem 1 Solution:

(1) If $f(x,y) = g(x)g(y)$ and
$$g(x) = (x+1)^2(x-1)^2 = (x^2-1)^2$$

then
$$g'(x) = 2 \cdot 2x\,(x^2-1) = 4x(x^2-1),$$
$$g''(x) = 4(x^2-1) + 8x^2 = 4(3x^2-1).$$

Thus,
$$\nabla f = \begin{bmatrix} g'(x)\,g(y) \\ g(x)\,g'(y) \end{bmatrix} = \begin{bmatrix} 4x(x^2-1)(y^2-1)^2 \\ 4y(y^2-1)(x^2-1)^2 \end{bmatrix}.$$

Now
$$\mathcal{H}_f(x,y) = \begin{bmatrix} g''(x)\,g(y) & g'(x)\,g'(y) \\ g'(x)\,g'(y) & g(x)\,g''(y) \end{bmatrix}$$
$$= \begin{bmatrix} 4(3x^2-1)(y^2-1)^2 & 16xy(x^2-1)(y^2-1) \\ 16xy(x^2-1)(y^2-1) & 4(x^2-1)^2(3y^2-1) \end{bmatrix}$$

Problem 1 Soln Cont...

(2) Matlab Code and output attached.

(3.) For $(0,0)$

$$H_f(0,0) = \begin{bmatrix} -12 & 0 \\ 0 & -\cancel{+}12 \end{bmatrix}$$

$$\nabla f = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore \quad x^{i+1} = x^{i} \quad \forall i \geq 0$ if

$x^{0} = (0,0)$.

Because H_f is negative definite $(0,0)$ is a local max.

```
function Df = Gradf(x)
Df = [4*x(1)*(x(1)^2 - 1)*(x(2)^2 - 1)^2; 4*x(2)*(x(2)^2 - 1)*(x(1)^2 - 1)^2];

function Hf = Hessf(x)
Hf = [4*(3*x(1)^2  - 1)*(x(2)^2 - 1)^2, 16*x(1)*x(2)*(x(1)^2 - 1)*(x(2)^2 - 1);
      16*x(1)*x(2)*(x(1)^2 - 1)*(x(2)^2 - 1), 4*(3*x(2)^2  - 1)*(x(1)^2 - 1)^2]
;

%v stores each iteration of our four 1x2 vectors
%we initiate these vectors now
v = zeros(4,2);
v(1,:)=[10,0];
v(2,:)=[0,-10];
v(3,:)=[-3,-2];
v(4,:)=[2,4];
%We set up a table that will store all of our
%values of the Newton iteration, and insert
%the initial condition
Tab = zeros(11,8);
Tab(1,:) = [v(1,:),v(2,:),v(3,:),v(4,:)];
for j=2:11
    %Update the vector with Newton's method
    for i=1:4
        v(i,:) = v(i,:) - (inv(Hessf(v(i,:)))*Gradf(v(i,:)))';
    end
    %Fill in a new line in the table
    Tab(j,:) = [v(1,:),v(2,:),v(3,:),v(4,:)];
end
%The columns of Tab store the iterations of v1,...,v4
v1 = Tab(:,1:2);
v2 = Tab(:,3:4);
v3 = Tab(:,5:6);
v4 = Tab(:,7:8);
%Print the table
table(v1,v2,v3,v4)
```

```
>> NewtonMethf

ans =

            v1                       v2                       v3                        v4
     _________________      ____________      ____________________      ________________

         10          0      0          -10          -3          -2          2          4
      6.689          0      0       -6.689     -2.5862     -1.7931     1.7988     3.4083
     4.4928          0      0      -4.4928     -2.2402     -1.6227     1.6334     2.9085
     3.0455          0      0      -3.0455     -1.9545     -1.4834      1.498     2.4905
      2.106          0      0       -2.106     -1.7221     -1.3703     1.3878     2.1453
     1.5181          0      0      -1.5181     -1.5366     -1.2795     1.2985     1.8649
     1.1832          0      0      -1.1832     -1.3916     -1.2075     1.2267     1.6414
     1.0353          0      0      -1.0353      -1.281     -1.1514     1.1694     1.4672
     1.0017          0      0      -1.0017     -1.1984     -1.1086     1.1245     1.3344
          1          0      0           -1     -1.1382     -1.0766     1.0899     1.2357
          1          0      0           -1     -1.0952     -1.0533     1.0638     1.1639

>>
```

18.085 PSET 3 Solutions

Problem 2

(1) We are solving $Ku = f$

where $K = \begin{pmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & 1 \end{pmatrix}$ fixed

and $f = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$.

Want to minimize total elongation $e_1 + e_2 + e_3 = u_3$.

Use MATLAB, one find

$f = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \Rightarrow u = \begin{pmatrix} 6 \\ 11 \\ 14 \end{pmatrix}$; $f = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \Rightarrow u = \begin{pmatrix} 6 \\ 11 \\ 13 \end{pmatrix}$

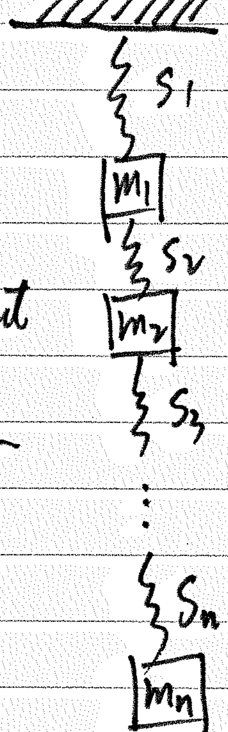
$f = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \Rightarrow u = \begin{pmatrix} 6 \\ 10 \\ 13 \end{pmatrix}$; $f = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \Rightarrow u = \begin{pmatrix} 6 \\ 10 \\ 11 \end{pmatrix}$

$f = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \Rightarrow u = \begin{pmatrix} 6 \\ 9 \\ 11 \end{pmatrix}$; $\boxed{f = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \Rightarrow u = \begin{pmatrix} 6 \\ 9 \\ 10 \end{pmatrix}}$

$u_3 = 10$ minimum

desired configuration!

(2)



n-object system

Pattern: If $m_1 \geq m_2 \geq \cdots \geq m_n$, one achieves the configuration w/ minimal total elongation.

Pf: Spring s_1 feels all n masses, its elongation $e_1 = (m_1 + \cdots + m_n)\, g/c$

spring s_2 detects $m_2, \ldots, m_n$, its elongation $e_2 = (m_2 + \cdots + m_n)\, g/c$

$\cdots$

Spring s_n feels only m_n, its elongation $e_n = m_n g/c$

Total elongation

$$= e_1 + \cdots + e_n = \frac{g}{c}\left(m_1 + 2m_2 + \cdots + n m_n\right)$$

To minimize it, we want $m_1 \geq m_2 \geq \cdots \geq m_n$.

Problem 3.

The system breaks if any of l_i satisfies

$$|l_i| > \frac{l}{2}, \quad \text{otherwise it is unbroken.}$$

Recall: $u = K^{-1} f = \frac{1}{4c}\begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} m_1 g \\ m_2 g \\ m_3 g \end{pmatrix}$

$$l = Au = AK^{-1}f = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \cdot \frac{g}{4c}\begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}$$

$$= \frac{g}{4c}\begin{pmatrix} 3 & 2 & 1 \\ -1 & 2 & 1 \\ -1 & -2 & 1 \\ -1 & -2 & -3 \end{pmatrix}\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}$$

$$= \frac{g}{4c}\begin{pmatrix} 3m_1 + 2m_2 + m_3 \\ -m_1 + 2m_2 + m_3 \\ -m_1 - 2m_2 + m_3 \\ -m_1 - 2m_2 - 3m_3 \end{pmatrix}$$

(1) As $m_i > 0$. We see that $|l_i| < \frac{3g}{4c}(m_1 + m_2 + m_3)$

for $i = 1, 2, 3, 4$

Therefore if $\frac{3g}{4c}(m_1+m_2+m_3) \leq \frac{l}{2}$, or equivalently

$$m_1+m_2+m_3 \leq \frac{2cl}{3g} = \frac{2}{3}M.$$

The system is always unbroken.

On the other hand

if $m_1+m_2+m_3 > \frac{2}{3}M = \frac{2cl}{3g}$

We may choose $m_2, m_3 \to 0^+$ very small

$$\text{s.t.} \quad m_1 > \frac{2cl}{3g}$$

$\Rightarrow \underline{\underline{(3m_1+2m_2+m_3)}} \cdot \frac{g}{4c} > \frac{l}{2}$, which breaks the system

So $\boxed{M' = \frac{2}{3}M.}$

(2) is very similar.

Notice we always have:

$$|l_1| = l_1 > |l_2|, |l_3|$$

$$|l_4| = -l_4 > |l_2|, |l_3|$$

So if the system breaks, either $|l_1| > \frac{l}{2}$ or $|l_4| > \frac{l}{2}$.

Notice $|l_1| + |l_4| = \frac{g}{4c}\left(3m_1 + 2m_2 + m_3 + m_1 + 2m_2 + 3m_3\right)$

$$= \frac{g}{c}(m_1 + m_2 + m_3).$$

So if $(m_1 + m_2 + m_3) > M$

then $|l_1| + |l_4| > \frac{g}{c} M = l$

either $|l_1| > \frac{l}{2}$ or $|l_4| > \frac{l}{2}$, the system breaks.

On the other hand, if $m_1 + m_2 + m_3 < M$, we may choose $m_1 = m_2 = m_3 < \frac{M}{3}$, then $|l_1| = |l_4| < \frac{l}{2}$, unbroken!

So $\boxed{M'' = M.}$

Problem 4

1) electronic circuits for instance

2) The solution is a linear combination of $\sin 2t$ and $\cos 2t$. Given the initial conditions $u(0)=1$ and $u'(0)=0$, we derive that $u(t)=\cos 2t$
Multiplying $(*)$ by $u'(t)$ and integrating, we have $u'(t)^2+4u^2=\text{cte}$
We can find the constant with the initial conditions: $u'(t)^2+4u^2(t)=4$.
In the phase plane, the parametric curve $t\mapsto(u(t),u'(t))$ describe an ellipse of equation $Y^2+4X^2=4$ and area $\pi\times 2\times 1=2\pi$
The period of $t\mapsto(\cos 2t,-2\sin 2t)$ is π. $\boxed{P=\pi}$

3) We denote by U_n (resp. V_n) the n^{th} approximation of u (resp. v) with time increment h. By definition the leapfrog operates as follows

$$\begin{cases} U_{n+1}=U_n+hV_n \\ V_{n+1}=V_n-4hU_{n+1}\end{cases} \Leftrightarrow \begin{cases} U_{n+1}=U_n+hV_n \\ V_{n+1}=-4hU_n+(1-4h^2)V_n\end{cases}$$

$$\begin{pmatrix}U_{n+1}\\V_{n+1}\end{pmatrix}=M(h)\begin{pmatrix}U_n\\V_n\end{pmatrix}\quad\text{with}\quad M(h)=\begin{pmatrix}1 & h\\ -4h & 1-4h^2\end{pmatrix}$$

$$\left(X^T\left(M\left(\tfrac{\pi}{N}\right)^N\right)X-1\right)N^3\xrightarrow[N\to\infty]{}\frac{\pi^3}{3}$$

4) $$X=\begin{pmatrix}1\\0\end{pmatrix}=\frac{1}{2}v_N^+ +\frac{1}{2}v_N^- +o(1)=\left(\frac{1}{2}+o(1)\right)v_N^+ +\left(\frac{1}{2}+o(1)\right)v_N^-$$

$$M\left(\tfrac{\pi}{N}\right)^N X=\left(\tfrac{1}{2}+o(1)\right)(\lambda_N^+)^N v_N^+ +\left(\tfrac{1}{2}+o(1)\right)(\lambda_N^-)^N v_N^-$$

$$X^T M\left(\tfrac{\pi}{N}\right)^N X=\left(\tfrac{1}{2}+o(1)\right)\left(1+\tfrac{2i\pi}{N}\right)^N\left(1+o(1)\right)+\left(\tfrac{1}{2}+o(1)\right)\left(1-\tfrac{2i\pi}{N}\right)^N\left(1+o(1)\right)$$

$$=\left(\tfrac{1}{2}+o(1)\right)\left(e^{2i\pi}+o(1)\right)\left(1+o(1)\right)+ \text{——}\left(e^{-2i\pi}+o(1)\right)\text{——}$$

$$=1+o(1).$$

Problem 5:

(1) Let $u(t) = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be a stationary solution of $\frac{du}{dt} = Au(t)$.

$\frac{du(t)}{dt} = 0$. Hence $A\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \iff \begin{cases} b = c \\ a = c \\ a = b \end{cases} \iff a = b = c$

$\Rightarrow$ the solution is stationary for initial conditions of the form $\begin{pmatrix} a \\ a \\ a \end{pmatrix}$

(2) A is skew-symetric: $A^T = -A$. Let us write $u(t) = \begin{pmatrix} a(t) \\ b(t) \\ c(t) \end{pmatrix}$

$$\|u(t)\|^2 = a^2(t) + b^2(t) + c^2(t)$$

$\frac{d}{dt}$

$$\frac{d}{dt}\|u(t)\|^2 = 2a(t)a'(t) + 2b(t)b'(t) + 2c(t)c'(t) = 2 \cdot u(t)^T \cdot \frac{du}{dt}$$
$$= 2 \cdot u(t)^T A u(t)$$
$$= 2\,(u(t)^T A u(t))^T$$
$$= 2\,(u(t)^T A^T (u(t))^T$$
$$= -2\, u(t)^T A u(t)$$
$$= -\frac{d}{dt}\|u(t)\|^2$$

$$\Rightarrow \frac{d}{dt}\|u(t)\|^2 = 0.$$

(3) $Q^T = I_3 + tA^T + \frac{t^2}{2}(A^T)^2 + \frac{t^3}{6}(A^T)^3 + \dots$

$(A^T = -A)$

$$= I_3 - tA + \frac{t^2}{2}A - \frac{t^3}{6}A^3 + \dots$$
$$= I_3 + (-t)A + \frac{(-t)^2}{2}A + \frac{(-t)^3}{6}A^3 + \dots$$
$$= e^{-tA}$$

tA and $-tA$ commute $\Rightarrow QQ^T = e^{tA} \cdot e^{-tA} = e^{tA - tA} = e^0 = I_3$

$$\frac{d}{dt}\left(e^{tA} \cdot X\right) = \frac{d}{dt}\left(tAX + \frac{t^2}{2}A^2X + \dots\right) = AX + tA^2X + \frac{t^2}{2}A^3X + \dots$$
$$= A\left(e^{tA}X\right).$$

define $u(t) = e^{tA} \cdot X$. Then $u(t)$ satisfies the equation and $u(0) = X$

Problem 6 Soln

(1) $a^T a \hat{u} = a^T b$

$a^T a = n \quad \Rightarrow \quad \boxed{\hat{u} = \frac{a^T b}{n} = \frac{\sum_{i=1}^{n} b_i}{n}}$

(2.) $e = b - a\hat{u}$

$$\Rightarrow \quad e^T e = \|e\|^2 = (b^T - a^T\hat{u})(b - a\hat{u})$$

$$= \|b\|^2 + \|a\|^2 \hat{u}^2 - 2 b^T a \hat{u}$$

$$= \sum_{i=1}^{n} b_i^2 + n\left(\frac{\sum_{i=1}^{n} b_i}{n}\right)^2 - 2\frac{\left(\sum_{i=1}^{n} b_i\right)^2}{n}$$

$$\boxed{\|e\|^2 = \sum_{i=1}^{n} b_i^2 - \frac{\left(\sum_{i=1}^{n} b_i\right)^2}{n}}$$

$$\therefore \quad \|e\| = \sqrt{\sum_{i=1}^{n} b_i^2 - \frac{\left(\sum_{i=1}^{n} b_i\right)^2}{n}}$$

(3.) Now

$$\boxed{\begin{aligned} &\left(\hat{b}\begin{pmatrix}1\\1\\1\end{pmatrix} - \begin{pmatrix}1\\2\\6\end{pmatrix}\right)(3,3,3) \\ &= 9\hat{b} - 27 = 0 \quad \text{if } \hat{b} = 3 \end{aligned}}$$

$p = \begin{pmatrix}3\\3\\3\end{pmatrix}$ is perpendicular

$$\boxed{\hat{b} = \frac{\sum_{i=1}^{3} b_i}{3} = \frac{9}{3} = 3}$$

$\leftarrow$ $\hat{b} = 3$ is closest

Now $\quad P = \vec{a}\,(a^T a)^{-1} a^T = \frac{1}{n} a a^T$

$$= \frac{1}{n}\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \\ 1 & \cdots & 1 \end{bmatrix}$$