Problem 1.

(1) $A = L \cdot U$ is invertible, therefore both L and U are as well.
$A = A^T$ reads $LU = U^T L^T$. Hence $(U^T)^{-1} L = L^T \cdot U^{-1}$ is both upper and lower triangular, i.e. diagonal. Let $D = U \cdot (L^T)^{-1}$.
Then $U = D \cdot L^T$ as we wanted, and $A = LDL^T$.

(2) In 1.3 terms in G. Strang's book, $A = LDL^T$ pos. definite means that all the coefficients of D are positive.

$$ {}^tX LDL^T X > 0 \quad \forall X \in \mathbb{R}^n \iff {}^tX D X > 0 \quad \forall X \in \mathbb{R}^n$$
(L is invertible)
$$\iff \sum_i d_i x_i^2 > 0 \quad \forall X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$$
$$\iff d_i > 0 \quad \forall i = 1 \dots n.$$

Problem 2

$$\Pi(a) = \begin{pmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{pmatrix}, \quad a \in \mathbb{R}.$$

* $\Pi(1) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, $\Pi(-2) = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$ are not invertible because their columns are linearly dependent.

* To get $\Pi(0)$'s first pivot, we need to switch row 1 and row 3.

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \Pi(0) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = LDL^T$$

* To get $\Pi(-1)$'s second pivot, we need to switch row 2 and row 3.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \Pi(-1) = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = LDL^T$$

for $a \notin \{0, 1, -1, -2\}$, $\Pi(a)$ admits a LU-factorization:

$$\Pi(a) = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1/a & 1 & 0 \\ 1/a & \frac{1}{a+1} & 1 \end{pmatrix}}_{L(a)} \underbrace{\begin{pmatrix} a & 0 & 0 \\ 0 & \frac{a^2-1}{a} & 0 \\ 0 & 0 & \frac{(a+2)(a-1)}{(a+1)} \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{a} & \frac{1}{a} \\ 0 & 1 & \frac{1}{a+1} \\ 0 & 0 & 1 \end{pmatrix}}_{U(a)}$$

18.085 PSet 2 Solutions:

Problem 3

(1) The general solutions to $-\frac{d^2u}{dx^2} = \delta(x-a)$ are of the form:

$$u(x) = -R(x-a) + Cx + D.$$

Impose the boundary conditions:

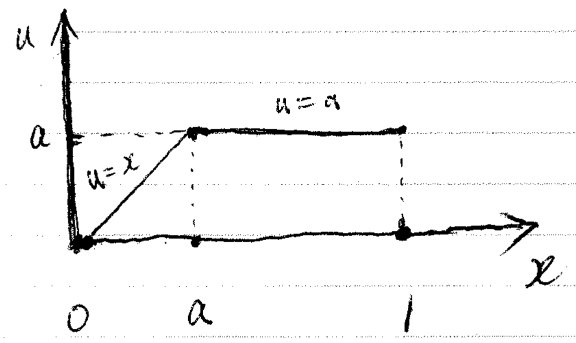
$u(0) = 0 \quad \Longrightarrow \quad D = 0.$

$u'(1) = 0 \quad \Longrightarrow \quad C = 1$

So the solution is

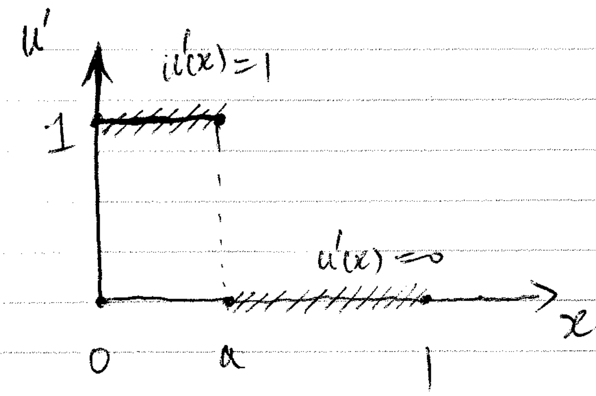
$$\boxed{u(x) = -R(x-a) + x}$$

(2). The graph of $u(x)$ is



The graph of $u'(x)$ is

(the "/////" part).



(3) To solve the discrete fixed-free problem, the matrix H to use is obtained from K by changing its lower right corner entry to 1.

i.e. $H_2 = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$, $H_3 = \begin{pmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & 1 \end{pmatrix}$, ...

if the distance between two adjacent nodes is $h = \frac{1}{n+1}$

the linear problem we are solving is

$$\frac{1}{h^2} H_n \cdot u = \frac{\delta}{h}.$$

The "h" below δ is crucial.

matlab solution:

$n=2$ $\quad u = \begin{pmatrix} 0.3333 \\ 0.3333 \end{pmatrix}$

$n=3$ $\quad u = \begin{pmatrix} 0.2500 \\ 0.2500 \\ 0.2500 \end{pmatrix}$

$n=4$ $\quad u = \begin{pmatrix} 0.2000 \\ 0.4000 \\ 0.4000 \\ 0.4000 \end{pmatrix}$

$n=5$ $\quad u = \begin{pmatrix} 0.1667 \\ 0.3333 \\ 0.3333 \\ 0.3333 \\ 0.3333 \end{pmatrix}$

The plot is ~~the~~ as follows

n=4
n=2 and n=5 coincide with analytical solution
n=3
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Problem 4

(1). $\delta'(f) = \int_{-\infty}^{\infty} \delta'(x) f(x)\, dx \overset{\text{formal integration by part}}{=} \int_{-\infty}^{\infty} (\delta(x) f(x))'\, dx$

$$- \int_{-\infty}^{\infty} \delta(x) f'(x)\, dx$$

$$= \underbrace{\delta(x) f(x) \Big|_{-\infty}^{+\infty}}_{0.} - \int_{-\infty}^{\infty} \delta(x) f'(x)\, dx$$

$$\overset{f \text{ vanishing at } \infty}{=} - \int_{-\infty}^{\infty} \delta(x) f'(x)\, dx$$

$$\overset{\text{by definition of } \delta}{=} \boxed{-f'(0)}$$

(2). The argument is almost identical to (1).

By our definition:

$$\delta'(f) = \lim_{n\to\infty} \int_{-\infty}^{\infty} f(x) \rho_n'(x)\, dx$$

$$\underset{\text{"genuine"}}{\overset{\text{integration by part}}{=}} \lim_{n\to\infty} \left(\int_{-\infty}^{\infty} (f(x)\rho_n(x))'\, dx - \int_{-\infty}^{\infty} f'(x) \rho_n(x)\, dx \right)$$

$$= \lim_{n\to\infty} \left(f(x)\rho_n(x) \Big|_{-\infty}^{+\infty} - \int_{-\infty}^{\infty} f'(x)\rho_n(x)\,dx \right.$$

$$\overset{f \text{ vanishing at } \infty}{=\!=} - \lim_{n\to\infty} \int_{-\infty}^{\infty} f'(x)\rho_n(x)\,dx$$

ρ_n approximate δ

$$= -\delta(f')$$

$$= \boxed{-f'(0).}$$

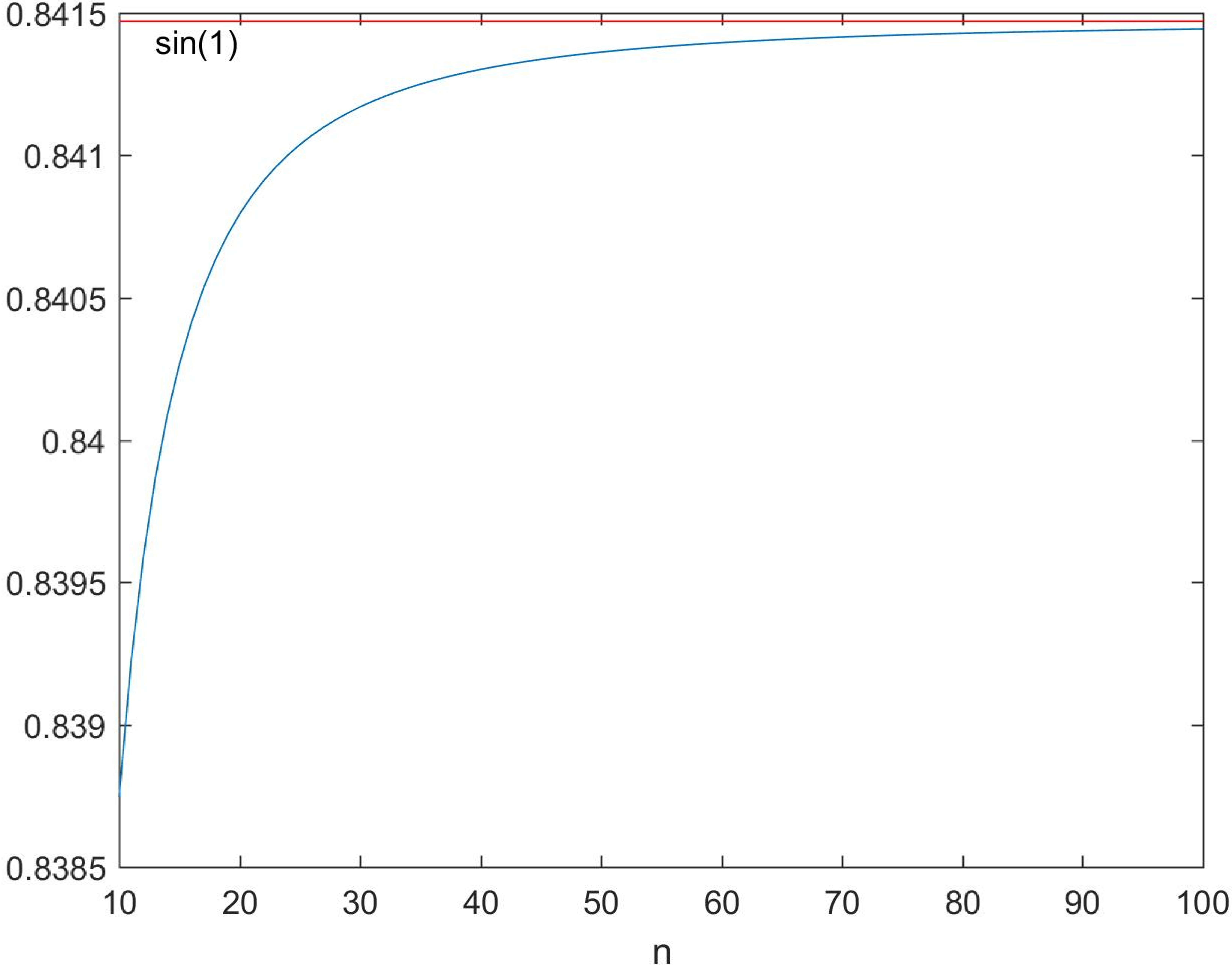
This definition has the advantage that the "integration by part" is genuine.

(3) We use MATLAB to check that

$$\int_{-\infty}^{\infty} \sin(\cos(x))\,\rho_n(x)\,dx \xrightarrow{n\to\infty} \sin(1) = 0.84147098\cdots$$

The plot of $\int_{-\infty}^{\infty} \sin(\cos(x))\,\rho_n(x)\,dx$ against n is below,

where the horizontal line is $y = \sin(1)$.



0.8415
0.841
0.8405
0.84
0.8395
0.839
0.8385
sin(1)
10
20
30
40
50
60
70
80
90
100
n

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Problem 5

- A and A^T have the same eigenvalues because

$$\boxed{\lambda \text{ is an eigenvalue of } A \text{ if and only if } \det(A - \lambda I) = 0}$$

Further

$$\boxed{\text{For any matrix } M, \ \det(M) = \det(M^T)}$$

therefore

$$\boxed{\det(A - \lambda I) = 0 \implies \det(A - \lambda I)^T = 0}$$

But

$$\boxed{(A - \lambda I)^T = A^T - \lambda I}$$

hence,

$$\boxed{\det(A^T - \lambda I) = 0}$$

whenever

$$\boxed{\det(A - \lambda I) = 0}$$

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Problem 5 Continued...

If A is a Markov Matrix, each row of A^T adds to 1. This implies that:

$$A^T \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} (A^T)_{1j} \\ \vdots \\ \sum_{j=1}^{n} (A^T)_{nj} \end{bmatrix}$$

but $\sum_{j=1}^{n} (A^T)_{ij}$ is precisely the i^{th} row sum of A^T, which is 1, therefore

$$A^T \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

so 1 is an eigenvalue of A^T because $\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ is its eigenvector.

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Problem 6

(1) If, $\sum_{i=1}^{n} \mu_i = 1$, and $\mu_i \geq 0$, we see

$$(Q\mu)_i = \sum_{j=1}^{n} Q_{ij}\mu_j$$

Each, $(Q\mu)_i \geq 0$, further, summing over i gives:

$$\boxed{\begin{aligned} \sum_{i=1}^{n} (Q\mu)_i &= \sum_{j=1}^{n}\left(\sum_{i=1}^{n} Q_{ij}\right)\mu_j \\ &= \sum_{j=1}^{n} \mu_j = 1 \end{aligned}}$$

where we have used

$$\cancel{\sum_{i=1}^{n} Q_{ij} = 1}$$

$$\boxed{\sum_{i=1}^{n} Q_{ij} = 1}$$

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(2.) First, note $Q = I_n - C_n/3$ where C_n is the circulant matrix.

Next, we can compute the eigenvalues of Q from those of C_n (in section 1.5) as follows

$$\boxed{\begin{aligned} & \det(\lambda I - Q) = 0 \\ \Rightarrow\ & \det(\lambda I - I + C_n/3) = 0 \\ \Rightarrow\ & \det((\lambda - 1) I + C_n/3) = 0 \\ \Rightarrow\ & (-1/3)^n \det(3(1-\lambda) I - C_n) = 0 \end{aligned}}$$

Now this means: ~~if~~ λ is an eigenvalue of Q, ~~then~~ if and only if $3(1-\lambda)$ is an eigenvalue of C_n. We know the eigenvalues of C_n (denoted by η_k):

$$\eta_k = 2 - 2\cos\left(\frac{2\pi k}{n}\right) \qquad k = 0, 1, \ldots, n-1$$

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Thus the eigenvalues of Q (called λ_k) satisfy:

$$\boxed{3(1-\lambda_k) = 2 - 2\cos\left(\frac{2\pi k}{n}\right) = \eta_k} \quad k=0,1,\dots,n-1$$

Rearranging gives

$$\boxed{\lambda_k = \frac{1}{3}\left(1 + 2\cos\left(\frac{2\pi k}{n}\right)\right)} \quad k=0,1,\dots,n-1$$

Now, we know

$$\boxed{Q^t = \sum_{i=1}^{n} \lambda_i^t \, v_i v_i^T} \quad \text{where}$$

$v_1, \dots, v_n$ are an orthonormal set of vectors (also the eigenvectors associated with λ_i).

all λ_i, $i \neq 1$ are less than 1.

Next, we see that all λ_i^t except λ_1^t go to 0. ($\lambda_1 = 1$) but remember $Q^t \vec{\mu} = \sum_{i=1}^{n} \lambda_i^t v_i v_i^T \vec{\mu}$

but $\vec{\mu} = \sum_{j=1}^{n} a_j v_j$, therefore,

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$$\boxed{\begin{aligned} Q^t \mu &= \sum_{i=1}^{n} a_i \lambda_i^t v_i \\ &\longrightarrow a_1 v_1 \end{aligned}}$$

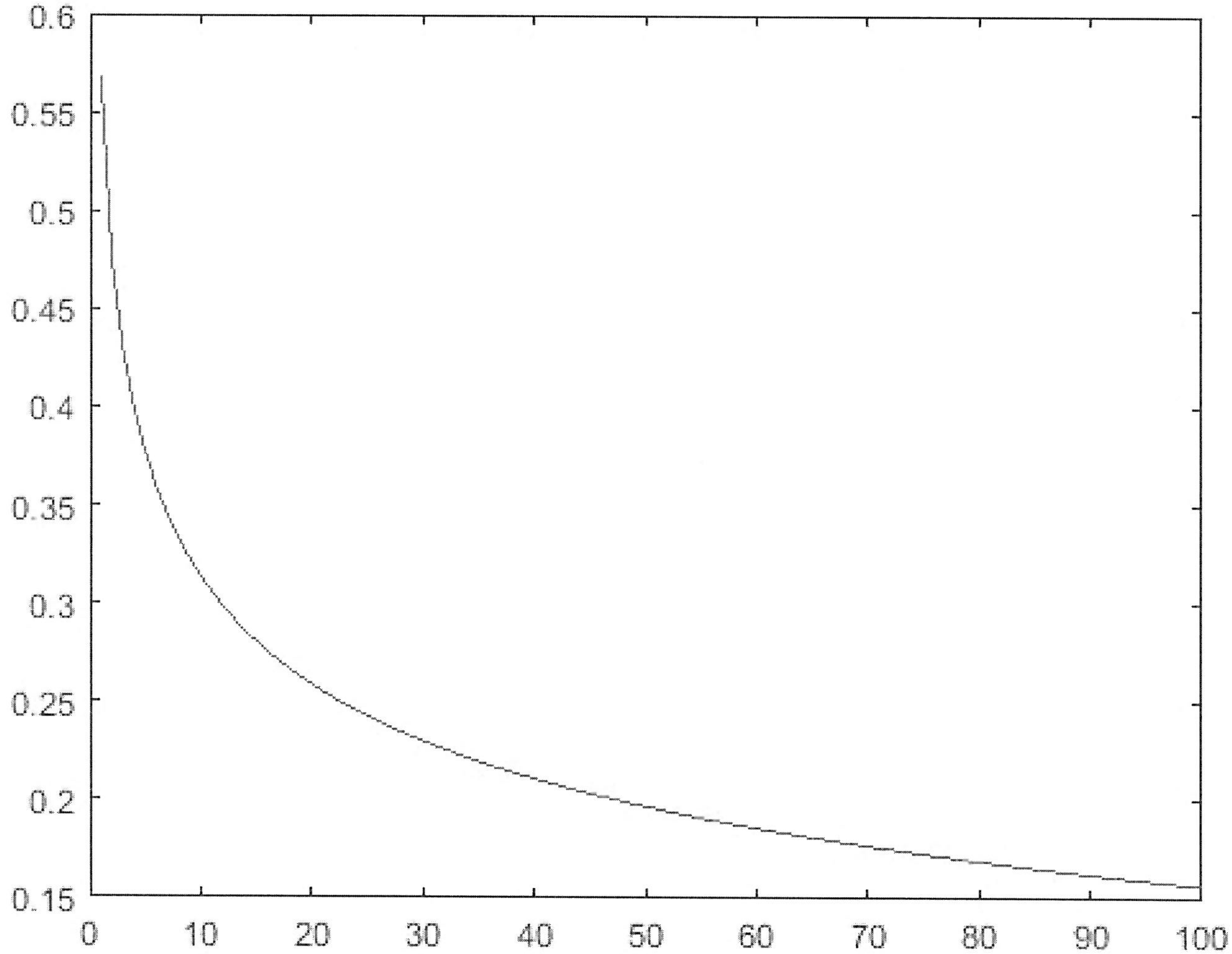
but by Problem 5, we know

$v_1 = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$, further by part

(1) we know $\sum_{j=1}^{n} (a_1 v_1)_j = 1$

$\therefore$ $Q^t \mu \longrightarrow \frac{1}{n} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ as claimed.

```
% Dimension of the matrix Q
n = 100;
% Length of time we run the chain T
T = 100;
% Create the transition matrix
Q = (1/3)*toeplitz([1,1,zeros(1,n-3),1]);
% Store the values of the L^2 norm from t=1, ..., T here
v = zeros(n,1);
% Store the time vector here
t = [1:T];
% Store the value of the L^2 norm of the jth iteration of the chain
% in the j-th component of v
for j=1:T
    v(j) = sqrt(sum(((Q^j)*[1,zeros(1,n-1)]' - (1/n)*ones(n,1)).^2));
end
% graph it.
plot(t,v);
```


0.6
0.55
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0
10
20
30
40
50
60
70
80
90
100