SOLUTIONS PSET 1

Q 1

a)

$$\left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[r_1^N = r_1/2]{} \left[\begin{array}{ccc|ccc} 1 & -1/2 & 0 & 1/2 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \rightarrow$$

"new"

$$\xrightarrow[r_2^N = r_2 + r_1]{} \left[\begin{array}{ccc|ccc} 1 & -1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 3/2 & -1 & 1/2 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{r_2^N = \frac{2}{3} r_2 \\ r_3^N = r_3 + \frac{2}{3} r_2}]{} \left[\begin{array}{ccc|ccc} 1 & -1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & -2/3 & 1/3 & 2/3 & 0 \\ 0 & 0 & 4/3 & 1/3 & 2/3 & 1 \end{array}\right] \rightarrow$$

$$\xrightarrow[\substack{r_1^N = r_1 + \frac{1}{2} r_2 \\ r_3^N = r_3 \frac{3}{4}}]{} \left[\begin{array}{ccc|ccc} 1 & 0 & -1/3 & 2/3 & 1/3 & 0 \\ 0 & 1 & -2/3 & 1/3 & 2/3 & 0 \\ 0 & 0 & 1 & 1/4 & 1/2 & 3/4 \end{array}\right] \xrightarrow[\substack{r_1^N = r_1 + \frac{1}{3} r_3 \\ r_2^N = r_2 + \frac{2}{3} r_3}]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/4 & 1/2 & 1/4 \\ 0 & 1 & 0 & 1/2 & 1 & 1/2 \\ 0 & 0 & 1 & 1/4 & 1/2 & 3/4 \end{array}\right]$$

$\underline{\underline{K_3^{-1}}}$

b) $\det(K_3) = (2\cdot 2\cdot 2) + \cancel{(-1)(-1)0} + \cancel{(-1)(-1)0} - \cancel{(0)2(0)} - (-1)(-1)2 - (-1)(-1)(2)$

$= 8 - 4 = \underline{\underline{4}}$

c) $\det(K_3^{-1}) = \dfrac{1}{\det(K_3)} = \underline{\underline{\dfrac{1}{4}}}$

Q 2

a)

$$C_4 = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & -1 & 0 & -1 \\ 0 & 3/2 & -1 & -1/2 \\ 0 & -1 & 2 & -1 \\ 0 & -1/2 & -1 & 3/2 \end{pmatrix} \longrightarrow$$

$l_{21} = l_{41} = -\frac{1}{2}$

$r_2^N = r_2 - l_{21} r_1$

$r_4^N = r_4 - l_{41} r_4$

$$\longrightarrow \begin{pmatrix} 2 & -1 & 0 & -1 \\ 0 & 3/2 & -1 & -1/2 \\ 0 & 0 & 4/3 & -4/3 \\ 0 & 0 & -4/3 & 4/3 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & -1 & 0 & -1 \\ 0 & 3/2 & -1 & -1/2 \\ 0 & 0 & 4/3 & -4/3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$r_3^N = r_3 - l_{32} r_2$

$l_{32} = -2/3$

$r_4^N = r_4 - l_{42} r_2$

$l_{42} = -1/3$

$r_4^N = r_4 - l_{43} r_3$

$l_{43} = -1$

$\underline{U} \Rightarrow \underline{L} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ -1/2 & -1/3 & -1 & 1 \end{pmatrix}$

b) We know C_4 is singular because the last element in U is 0.

I.E. - one of the pivots is zero

- the determinant $\left(2 \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot 0\right)$ is zero

c) The last column of U as new non-zeros because the top right -1 from C_4 propagates downwards with the row elimination operations.

Q 3

a) By rows

$$\underset{(2\times 2)}{\overset{A}{\begin{pmatrix}2&3\\4&5\end{pmatrix}}}\underset{(2\times 1)}{\overset{x}{\begin{pmatrix}1\\2\end{pmatrix}}}=\begin{pmatrix}row_1^A\cdot x\\ row_2^A\cdot x\end{pmatrix}=\begin{pmatrix}2\cdot 1+3\cdot 2\\ 4\cdot 1+5\cdot 2\end{pmatrix}=\underline{\underline{\begin{pmatrix}8\\14\end{pmatrix}}}$$

By columns

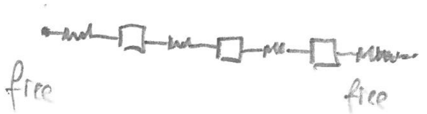
$$\begin{pmatrix}2&3\\4&5\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix}=col_1^A\cdot x_{11}+col_2^A\cdot x_2=1\begin{pmatrix}2\\4\end{pmatrix}+2\begin{pmatrix}3\\5\end{pmatrix}=\begin{pmatrix}2\\4\end{pmatrix}+\begin{pmatrix}6\\10\end{pmatrix}$$

$$=\begin{pmatrix}8\\14\end{pmatrix}$$

b) The nullspace of B_3 contains the all constant vector ⇒ Solution will be $\begin{pmatrix}0\\0\\0\end{pmatrix}$

By Rows

$$\underset{B_3}{\begin{pmatrix}1&-1&0\\-1&2&-1\\0&-1&1\end{pmatrix}}\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}\text{inner prod usin row}_1\\ \text{" " " row}_2\\ \text{" " " row}_3\end{pmatrix}=$$



$$=\begin{pmatrix}(1\ -1\ \ 0)\cdot(1\ \ 1\ \ 1)\\(-1\ \ 2\ -1)\cdot(1\ \ 1\ \ 1)\\(0\ -1\ \ 1)\cdot(1\ \ 1\ \ 1)\end{pmatrix}=\begin{pmatrix}1-1+0\\-1+2-1\\0-1+1\end{pmatrix}=\underline{\underline{\begin{pmatrix}0\\0\\0\end{pmatrix}}}$$

By Colunns

$$\underbrace{1\begin{pmatrix}1\\-1\\0\end{pmatrix}+1\begin{pmatrix}-1\\2\\-1\end{pmatrix}+1\begin{pmatrix}0\\-1\\1\end{pmatrix}}_{\text{conbination of colums}}=\underline{\underline{\begin{pmatrix}0\\0\\0\end{pmatrix}}}$$

Q ⑤ $$\frac{-u_2 + 8u_1 - 8u_{-1} + u_{-2}}{12h} = \frac{du}{dx} + 6h^4 \frac{d^5u}{dx^5} + \cdots$$

$\boxed{1}$

$\rightarrow$ $u = 1$ $\rightarrow$ $\frac{du}{dx} = 0$

$$\frac{du}{dx} = \frac{-1 + 8(1) - 8(1) + 1}{12h} = 0 \quad \text{Correct}$$

$\rightarrow$ $u = x^2$ $\rightarrow$ $\frac{du}{dx} = 2x$

$$\frac{du}{dx} = \frac{-(x+2h)^2 + 8(x+h)^2 - 8(x-h)^2 + (x-2h)^2}{12h} =$$

$$= \frac{8\left[(x+h)^2 - (x-h)^2\right] + (x-2h)^2 - (x+2h)^2}{12h} =$$

$$= \frac{8\left[(x+h) + (x-h)\right]\left[(\cancel{x}+h) - (x-h)\right] + \left[(x-2h) + (x+2h)\right]\left[(\cancel{x}-2h) - (x+2h)\right]}{12h}$$

$$= \frac{8(2x)(2h) + (2x)(-4h)}{12h} = \frac{32x\cancel{h} - 8x\cancel{h}}{12\cancel{h}} = \frac{24x}{12} = 2x \quad \text{Correct}$$

$\rightarrow$ $u = x^4$ $\rightarrow$ $\frac{du}{dx} = 4x^3$

$$\frac{du}{dx} = \frac{-(x+2h)^4 + 8(x+h)^4 - 8(x-h)^4 + (x-2h)^4}{12h} =$$

$$= \frac{1}{12h}\left\{\left[(x-2h)^2 + (x+2h)^2\right]\left[(\cancel{x}-2h)^2 - (x+2h)^2\right] + 8\left[(\cancel{x}+h)^2 + (x-h)^2\right]\left[(\cancel{x}+h)^2 - (x-h)^2\right]\right\} =$$

$$= \frac{1}{12h}\left\{\left[x^2 + 4h^2 - \cancel{4hx} + x^2 + 4h^2 + \cancel{4hx}\right]\left[\cancel{x^2} + \cancel{4h^2} - 4hx - \cancel{x^2} - \cancel{4h^2} - 4hx\right] + 8\left[x^2 + h^2 + \cancel{2hx} + x^2 + h^2 - \cancel{2hx}\right]\left[\cancel{x^2} + \cancel{h^2} + 2hx - \cancel{x^2} - \cancel{h^2} + 2hx\right]\right\}$$

$$= \frac{1}{12h}\left[(-8hx)\left(2x^2+8h^2\right) + 8(4hx)\left(2x^2+2h^2\right)\right] =$$

$$= \frac{1}{12\cancel{h}}\left[-16\cancel{h}x\left(x^2+4h^2\right) + 4\left(16\cancel{h}x\right)\left(x^2+h^2\right)\right] =$$

$$= \frac{4}{3}x\left[-\left(x^2+4h^2\right)+4\left(x^2+h^2\right)\right] = \frac{4}{\cancel{3}}x\left(\cancel{3}x^2\right) = \underline{\underline{4x^3}} \quad \text{Correct.}$$

$\boxed{2}$ (a) $u(x+h) = u(x) + h\,u'(x) + \frac{h^2}{2}u''(x) + \frac{h^3}{6}u'''(x) + \frac{h^4}{24}u^{IV}(x) + \frac{h^5}{120}u^{V}(x) + \dots$

(b) $u(x-h) = u(x) - (\text{same}) + (\text{same}) - (\text{same}) + (\text{same}) - (\text{same}) + \dots$

(c) $u(x+2h) = u(x) + 2h\,u'(x) + \frac{(2h)^2}{2}u''(x) + \frac{(2h)^3}{6}u'''(x) + \frac{(2h)^4}{24}u^{IV}(x) + \frac{(2h)^5}{120}u^{V}(x) \dots$

(d) $u(x-2h) = u(x) - (\text{same}) + (\text{same}) - (\text{same}) + (\text{same}) - (\text{same}) + \dots$

8(a) − 8(b) − (c) + (d): $8u_1 - 8u_{-1} - u_2 + u_{-2} = \emptyset + 12h\,u'(x) + \emptyset + \underbrace{(8+8-8-8)}_{\emptyset}\frac{h^3}{6}u''' + \underbrace{(8-8-16+16)}_{\emptyset}\frac{h^4}{24}u^{IV}(x) +$

$$+ \underbrace{(8+8-32-32)}_{-48}\frac{h^5}{120}u^{V}(x) + \dots$$

$$\Rightarrow -u_2 + 8u_1 - 8u_{-1} - u_{-2} = 12h\,\underline{\underline{u'(x)}} - \left(\frac{2}{5}\right)\cdot h^5 u^{V}(x) + \dots$$

$$\frac{-u_2 + 8u_1 - 8u_{-1} - u_{-2}}{12h} = u'(x) - \frac{1}{30}h^4 u^{V}(x) + \dots$$

↳ $\underline{\underline{\text{Coeff } b}}$

⑥ $$\frac{d^3u}{dx^3} \approx \frac{u_{i+2} - 2u_{i+1} + 2u_{i-1} - u_{i-2}}{2h^3}$$ Easier than Q5 !!

Same Taylor expansion for u_{i+2}; u_{i+1}, u_{i-1}, u_{i-2}

Combining them

$$\frac{u_{i+2}-2u_{i+1}-2u_{i-1}-u_{i-2}}{2h^3} = \frac{1}{2h^3}\Big[\underbrace{\cancel{(1-2+2+1)}}_{2} u(x) + \overset{\emptyset}{\cancel{(2-2-2+2)}}\, h\, u'(x) + \overset{\emptyset}{\cancel{\left(\frac{2^2}{2} - \frac{2}{2} + \frac{2}{2} - \frac{2^2}{2}\right)}} h^2 u''(x)$$

$$+ \left(\frac{2^3}{6} - \frac{2}{6} - \frac{2}{6} + \frac{2^3}{6}\right) h^3 u'''(x) + \overset{\emptyset}{\cancel{\left(\frac{2^4}{24} - \frac{2}{24} + \frac{2}{24} - \frac{2^4}{24}\right)}} h^4 u^{IV}(x)$$

$$+ \underbrace{\left(\frac{2^5}{120} - \frac{2}{120} - \frac{2}{120} + \frac{2^5}{120}\right)}_{1/2} h^5 u^{V}(x) + \cdots \Big]$$

$$= u'''(x) + \frac{1}{4} h^2 u^{V}(x) + \cdots$$

$$\Rightarrow \boxed{u'''(x) = \frac{u_{i+2} - 2u_{i+1} - 2_{i-1} - u_{i-2}}{2h^3} - \frac{1}{4} h^2 u^{V}(x) + \cdots}$$

$O(h^2)$ Second order accuracy.