Solution to problem 1.

(a) $w(x,y) = (x^2-y^2, 2xy)$

$$\operatorname{curl} w(x,y) = \frac{\partial}{\partial y}(x^2-y^2) - \frac{\partial}{\partial x}(2xy)$$

$$= -2y - 2y = -4y \neq 0$$

So $w(x,y)$ is <u>not</u> a gradient vector field.

$$\operatorname{div} w(x,y) = \frac{\partial}{\partial x}(x^2-y^2) + \frac{\partial}{\partial y}(2xy)$$

$$= \underline{4x}$$

If (u,s) is a C-R pair

then
$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial s}{\partial y} & ① \\[2ex] \dfrac{\partial s}{\partial x} = -\dfrac{\partial u}{\partial y} & ② \end{cases}$$

$\Rightarrow$ (s,u) is a gradient

So $\operatorname{curl}(s,u) = \dfrac{\partial s}{\partial y} - \dfrac{\partial u}{\partial x} = 0$ from ①

$\operatorname{div}(s,u) = \dfrac{\partial s}{\partial x} + \dfrac{\partial u}{\partial y} = 0$ from ②.

(b). $f(x+iy) = x+iy + \dfrac{1}{x+iy}$

$= x+iy + \dfrac{x-iy}{x^2+y^2}$

$= \left(x + \dfrac{x}{x^2+y^2}\right) + i\left(y - \dfrac{y}{x^2+y^2}\right)$

So $\mathrm{Re}(f) = x + \dfrac{x}{x^2+y^2}$

and $\mathrm{Im}(f) = y - \dfrac{y}{x^2+y^2}$ are solutions to Laplace eqn.

In polar coordinates:

$\mathrm{Re}(f) = r\cos\theta\left(1 + \dfrac{1}{r^2}\right) = \left(r + \dfrac{1}{r}\right)\cos\theta$

$\mathrm{Im}(f) = r\sin\theta\left(1 - \dfrac{1}{r^2}\right) = \left(r - \dfrac{1}{r}\right)\sin\theta$

(c) For $u=1$, $$\int_C d\theta = \int_0^{2\pi} d\theta = 2\pi$$

For $u = r\cos\theta$,

$$\int_C u\,d\theta = \int_0^{2\pi} \cos\theta\,d\theta = 0$$

For $u = r^2\cos 2\theta$,

$$\int_C u\,d\theta = \int_0^{2\pi} \cos 2\theta\,d\theta = 0.$$

To use divergence thm, notice

$$u\,d\theta = u(\vec{n}\cdot\vec{n})\,d\theta \,.$$

hence $$\int_C u\,d\theta = \int_D \operatorname{div}(u\vec{n})\,dxdy = \int_D \operatorname{div}(u(x,y))\,dxdy$$

D ← unit disk

For $u=1$

$$\int_C u\,d\theta = \int_C (x,y)\cdot\vec{n}\,d\theta = \int_D \operatorname{div}(x,y)\,dxdy$$

$$= 2\int_D dx\,dy \quad = 2\pi$$

For $u = r\cos\theta = x$

$$\int_C u\,d\theta = \int_D \operatorname{div}(x^2, xy)\,dx\,dy$$

$$= \int_D 3x\,dx\,dy \quad = 0 \quad \text{by symmetry } x \leftrightarrow -x$$

For $u = r^2\cos 2\theta = x^2 - y^2$

$$\int_C u\,d\theta = \int_D \operatorname{div}(x^3 - xy^2, x^2y - y^3)\,dx\,dy$$

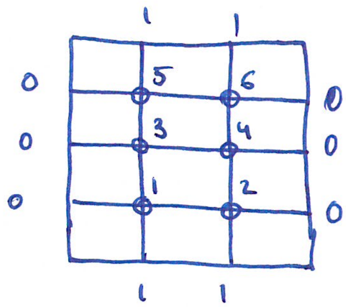
$$= 4\int_D (x^2 - y^2)\,dx\,dy$$

$$= 0 \quad \text{by symmetry } (x \leftrightarrow y)$$

Problem 2 (35 points):
Solve Poisson's equation with $f(x, y) = 1$ on a square of side length 1 by *finite differences* considering two *interior* nodes in the x direction and three *interior* nodes in the y direction. The boundary conditions are:

- $u = 1$ along the horizontal sides (x direction) including the corners of the square
- $u = 0$ along the vertical sides (y direction) excluding the corners of the square

** Do not compute the actual solution. Just write down the matrices involved in the calculation.



$$\left\{ \begin{array}{l} \Delta x = \frac{1}{3} \\ \Delta y = \frac{1}{4} \end{array} \right.$$

$$-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} \approx \frac{-u_{i-1,j} + 2u_{ij} - u_{i+1,j}}{\Delta x^2} + \frac{-u_{i,j+1} + 2u_{ij} - u_{i,j-1}}{\Delta y^2}$$

Row 1

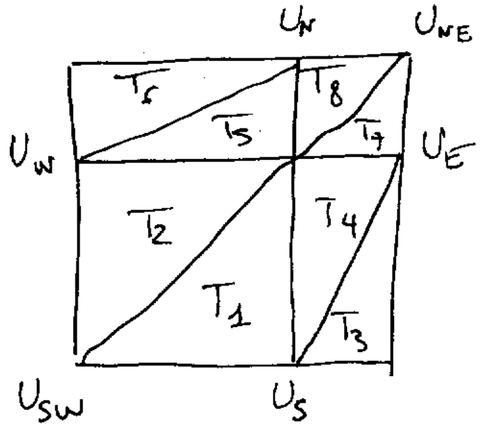
$$\frac{-\cancel{u_{1W}}^{\,0} + 2u_1 - u_2}{\Delta x^2} + \frac{-u_{1S} + 2u_1 - u_3}{\Delta y^2} = \underbrace{2C}_{\boxed{C = \frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}}} u_1 - \frac{1}{\Delta x^2} u_2 - \frac{1}{\Delta y^2} u_3 - \frac{\overset{1}{u_{1S}}}{\Delta y^2} = 1$$

$$\Rightarrow \quad 2Cu_1 - \frac{1}{\Delta x^2} u_2 - \frac{1}{\Delta y^2} u_3 = 1 + \frac{1}{\Delta y^2} \quad \text{(Row 1)}$$

Similarly:

$$\overbrace{\begin{pmatrix} 2C & -\frac{1}{\Delta x^2} & -\frac{1}{\Delta y^2} & 0 & 0 & 0 \\ -1/\Delta x^2 & 2C & 0 & -1/\Delta y^2 & 0 & 0 \\ -1/\Delta y^2 & 0 & 2C & -1/\Delta x^2 & -1/\Delta y^2 & 0 \\ 0 & -1/\Delta y^2 & -1/\Delta x^2 & 2C & 0 & -1/\Delta y^2 \\ 0 & 0 & -1/\Delta y^2 & 0 & 2C & -1/\Delta x^2 \\ 0 & 0 & 0 & -1/\Delta y^2 & -1/\Delta x^2 & 2C \end{pmatrix}}^{K2D} \overbrace{\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix}}^{u} = \overbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}}^{F} + \overbrace{\begin{pmatrix} 1/\Delta y^2 \\ 1/\Delta y^2 \\ 0 \\ 0 \\ 1/\Delta y^2 \\ 1/\Delta y^2 \end{pmatrix}}^{\text{Boundary Cond.}}$$

Problem 3:



We approximate the solution of $\Delta u = -1$ with a piecewise linear functions ~~ϕ_i~~ on each triangle T_i. The only unknown is the value of ~~these~~ functions at $\left(\frac{2}{3}, \frac{2}{3}\right)$ so we need one trial function ϕ and one equation. ϕ is also piecewise linear on every T_i, worths 0 on the boundary and 1 at $\left(\frac{2}{3}, \frac{2}{3}\right)$.

$$\iint_{\text{square}} \left(\phi_x \Psi_x + \phi_y \Psi_y\right) dx\,dy = \iint_{\text{square}} \phi \, dx\,dy$$

Note that $\phi_x \Psi_x + \phi_y \Psi_y$ is piecewise constant and worths 0 on T_6 and T_3.

$$\iint_{T_1} \phi_x \Psi_x + \phi_y \Psi_y = \frac{2}{9} \cdot \left(\frac{9}{4}(U-1)\right)$$

$$\iint_{T_2} \quad " \quad = \frac{2}{9} \left(\frac{9}{4}(U-1)\right)$$

$$\iint_{T_4} \quad " \quad = \frac{1}{9} \left(\left(9 + \frac{9}{4}\right)(U-1)\right)$$

$$\iint_{T_5} \quad " \quad = \frac{1}{9} \left(\left(9 + \frac{9}{4}\right)(U-1)\right)$$

$$\iint_{T_7} \quad " \quad = \frac{1}{18} \left(9(U-1)\right)$$

$$\iint_{T_8} \quad " \quad = \frac{1}{18} \left(9(U-1)\right)$$

$$\Rightarrow \iint_{\text{square}} \left(\phi_x \Psi_x + \phi_y \Psi_y\right) dx\,dy = \frac{9}{2}(U-1) = \frac{1}{3}\left(1 - \text{area}(T_6) - \text{area}(T_3)\right)$$

$$= \frac{1}{3}\left(\frac{7}{9}\right)$$

$$\boxed{U = \frac{257}{243}}$$