Exam 2

November 5, 2015

Time limit: 80 minutes

Name (Print):

This exam contains 10 pages (including this cover page) and 4 problems. Check to see if any pages are missing.

Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use your course textbook and notes. No electronic devices or other materials are allowed.

You are required to show your work on each problem on this exam, unless otherwise specified. The following rules apply:

- **Organize your work**, in a neat and coherent way. Work without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive very little credit; an incorrect answer supported by substantially correct calculations and explanations will still receive partial credit.
- $\boxed{\text{Box}}$ your final answer to each problem.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	20	
2	30	
3	20	
4	30	
Total:	100	
	Total (%):	

Problem 1 (20 points):
Set up the problem to fit a parabola $\hat{u} = C + Dt + Et^2$ through these (t, b) points, minimizing the sum of the squares of the error:

$$(0,1),\ (1,1),\ (2,4). \tag{1}$$

(a) (8 points) Give the matrix A and the vector b for the least squares formulation of the problem.

$t=0 \rightarrow C + D0 + E0 = 1$

$t=1 \rightarrow C + D1 + E1 = 1$

$t=2 \rightarrow C + D2 + E4 = 1$

$$\downarrow$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} C \\ D \\ E \end{pmatrix}}_{u} = \underbrace{\begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}}_{b}$$

(b) (8 points) Calculate A^TA and A^Tb, and give the Matlab command that would give you the coefficients of the fitted parabola.

$$A^TA\hat{u} = A^Tb$$

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{pmatrix}}_{A^T}\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} 3 & 3 & 5 \\ 3 & 5 & 9 \\ 5 & 9 & 17 \end{pmatrix}}_{A^TA}$$

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{pmatrix}}_{A^T}\underbrace{\begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}}_{b} = \underbrace{\begin{pmatrix} 6 \\ 9 \\ 17 \end{pmatrix}}_{A^Tb}$$

Matlab command

$u = (A^TA) \backslash (A^Tb)$

Solution $u = \begin{bmatrix} 1 \\ -3/2 \\ 3/2 \end{bmatrix}$

(c) (4 points) What is the minimum value $E = e_1^2 + e_2^2 + e_3^2$?

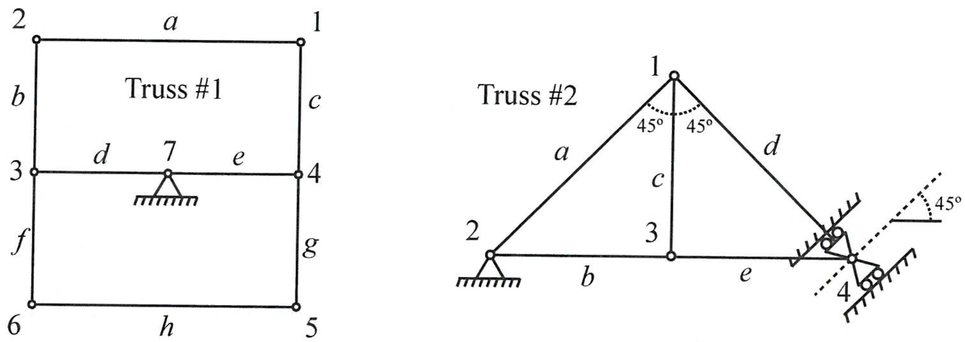
Three points define a parabola $\Rightarrow$ E will be $\underline{0}$

(Not necessary)

$$e = b - A\bar{u} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ -3/2 \\ 3/2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

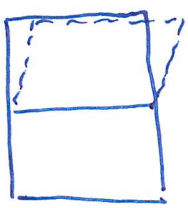
$$E = e_1^2 + e_2^2 + e_3^2 = \underline{0}$$

Problem 2 (30 points):
Consider the following trusses with numbered nodes lettered edges:

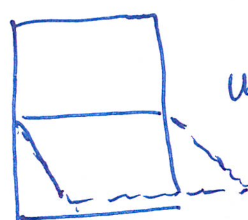


(a) (10 points) Is Truss #1 stable? If not, identify the mechanisms and rigid motions and provide the corresponding u's that conform the null space of the incidence matrix A.

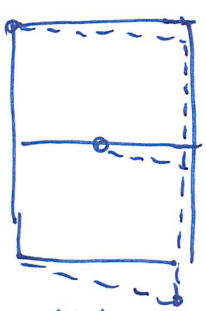
Unknowns $= 2N - R = 2 \cdot 7 - 2 = 12$
Bars $= 8$
$\Longrightarrow 12 - 8 = 4$ mechanisms or rigid motions


Mech #1

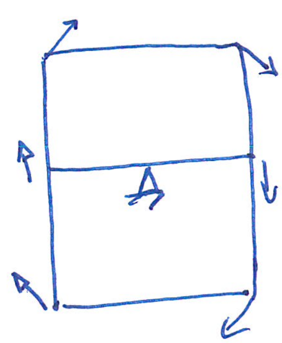
$u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ ✓


Mech #2

$u_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ ✓

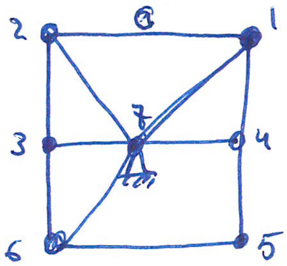

Mech #3

$u_3 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ ✓


Rigid motion

$u_{RM} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ -1 \\ -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$ ✓

(b) (5 points) In Truss #1, add (draw) the minimum number of bars necessary to block all the mechanisms.



$\rightarrow$ Rank(A) = 11

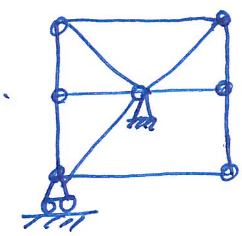
Fourth bar is not necessary

(c) (5 points) Following from the previous question, what is the rank of the new matrix A? Guess and explain your answer. Do not write the new incidence matrix.

A would be 11x12. Adding more bars blocks the mechanisms but not the rigid motion. Rank(A) = 11

We need to add at least one constraint with a support to make the truss stable

e.g.



$\Rightarrow$ New A would be 11x11 with Rank(A) = 11

(d) (10 points) Now, let's consider Truss #2. Note that in node 4 the pin joint is mounted on a roller support, which allows node displacements parallel to rolling surface only. Is this truss stable? If not, draw the mechanisms and rigid motions, and provide the corresponding vector displacements, i.e the u's. Write the matrix A whose null space explains your answer. You can guess the rank.

We have six unknowns $(u_1^H, u_1^V, u_3^H, u_3^V, u_4^H, u_4^V)$

$\Rightarrow$ If the truss is stable, the rank of A should be 6

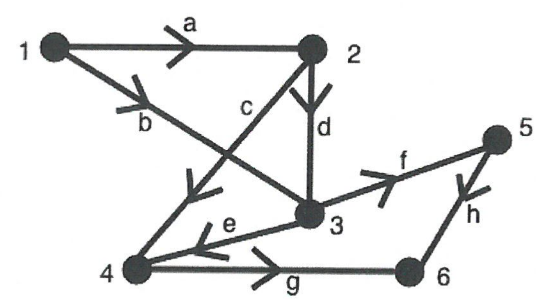
We only have 5 bars, which may make it look impossible, however we should note that $u_4^V = u_4^H$

$$
\begin{array}{ll|c}
 & & \begin{array}{cccccc} u_1^H & u_1^V & u_3^H & u_3^V & u_4^H & u_4^V \end{array} \\
\text{bars} & \begin{array}{c} a \\ b \\ c \\ d \\ e \\ u_4^H - u_4^V = 0 \rightarrow \end{array} & \underbrace{\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}}_{A}
\end{array}
\begin{pmatrix} u \end{pmatrix} = 0
$$

Rank(A) = 6 ✓

Problem 3 (20 points):

Consider the following electrical circuit with six nodes (numbered) and eight edges (lettered):



(a) (3 points) Write down the 8×6 incidence matrix A.

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \end{array}$$

(b) (5 points) Describe the vectors x such that $Ax = 0$ and the vectors w such that $A^T w = 0$..

$Ax = 0$

$$x = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$A^T w = 0$ (Loops)

$$w_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad w_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad w_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

(c) (3 points) Give the physical interpretation of $A^T w = 0$, where $w = (0, 0, 1, -1, 0, -1, 1, -1)^T$.

KCL around the loop 2-4-6-5-3-2 with no. external sources of current.

(d) (3 points) Let A be the incidence matrix, e the vector of potential differences, and w the vector of edge currents. State Kirchhoff's current and voltage law and as relations involving A, e, and w. (You may like to write down the laws in words first.)

KCL $A^T w = 0$ supposing no external sources.

KVL $e^T w = 0$

(e) (3 points) Show the following for vectors x, y, and z: If $y = Ax$ and $A^T z = 0$, then $y^T z = 0$.

$\begin{cases} y = Ax \\ A^T z = 0 \end{cases}$ $y^T = (Ax)^T$

$y^T z = (Ax)^T z = x^T \underbrace{A^T z}_{= 0} = x^T 0 = 0$ ✓

(f) (3 points) Explain, using the result from the previous question, why Kirchhoff's current implies Kirchhoff's voltage law.

Let $z = w$, $y = e$, $x =$ pot. differences

By (e) $e = Ax$ (by definition)

$A^T w = 0 \Rightarrow e^T w = 0$

Current law $\qquad$ Voltage law

Problem 4 (30 points):
Set up the finite element method for the equation

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(c(x)\frac{\mathrm{d}u}{\mathrm{d}x}\right) = \delta(x-2/3), \qquad 0 \le x \le 1 \tag{2}$$

where

$$c(x) = \begin{cases} 2 & \text{if } x < 1/3 \\ 4 & \text{if } x > 1/3 \end{cases} \tag{3}$$

We take boundary conditions

$$w(0) = u(1) = 0 \tag{4}$$

where $w(x) = c(x)\frac{\mathrm{d}u}{\mathrm{d}x}$.

(a) (5 points) Write the weak form of the differential equation. What conditions must the test function $v(x)$ satisfy?

$$\int_0^1 -\frac{d}{dx}\left(c(x)\frac{du}{dx}\right)v(x)\,dx = \int_0^1 \delta(x-2/3)\,v(x)\,dx$$

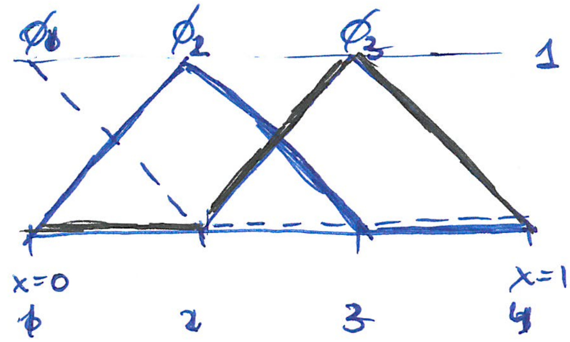
LHS integrated by parts

$$\int_0^1 -\frac{d}{dx}\left(c(x)\frac{du}{dx}\right)v(x)\,dx = \left[c(x)\frac{du}{dx}v(x)\right]_0^1 + \int_0^1 c(x)\frac{du}{dx}\frac{dv}{d}\,dx$$

BCs make this term 0
The test function satisfies $v(1)=0$

$$\Rightarrow \boxed{\int_0^1 c(x)\frac{du}{dx}\frac{dv}{dx}\,dx = \int_0^1 \delta(x-2/3)\,v(x)\,dx}$$

(b) (5 points) Take $h = 1/3$. Draw the hat functions that you will use to solve this problem.



(c) (10 points) Construct the matrix K for this problem.

$$K_{ij} = \int_0^1 c(x)\, \phi_i'(x)\, \phi_j'(x)\, dx$$

ϕ's = U's → The derivatives are ± 3

$c(x)$ changes at $x = 1/3$

$$K_{11} = \int_0^1 c(x) (\phi_1'(x))^2 dx = \int_0^{1/3} 2 \cdot (-3)^2 dx = 6$$

$$K_{22} = \int_0^1 c(x) (\phi_2')^2 dx = \int_0^{1/3} 2 \cdot 3^2 dx + \int_{1/3}^{2/3} 4 \cdot (-3)^2 dx = 18$$

$$K_{33} = \int_0^1 c(x) (\phi_3')^2 dx = \int_{1/3}^{2/3} 4 \cdot 3^2 dx + \int_{2/3}^1 4\, (-3)^2 dx = 24$$

$$K_{12} = K_{21} = \int_0^1 c(x)\, \phi_1'\, \phi_2'\, dx = \int_0^{1/3} 2 \cdot 3 \cdot (-3)\, dx = -6$$

$$K_{23} = K_{32} = \int_0^1 c(x)\, \phi_2' \phi_3'\, dx = \int_{1/3}^{2/3} 4\, (-3)\, 3\, dx = -12$$

$$K_{13} = K_{31} = \int_0^1 c(x)\, \phi_1' \phi_3' dx = 0$$

$$K = \begin{pmatrix} 6 & -6 & 0 \\ -6 & 18 & -12 \\ 0 & -12 & 24 \end{pmatrix}$$

(d) (5 points) Construct the vector F for this problem.

$$F_i = \int_0^1 \delta(x - 2/3)\, \phi_i(x)\, dx$$

$$F_1 = \int_0^1 \delta(x - 2/3)\, \phi_1\, dx = \phi_1(2/3) = 0$$

$$F_2 = \int_0^1 \delta(x - 2/3)\, \phi_2\, dx = \phi_2(2/3) = 0$$

$$F_3 = \int_0^1 \delta(x - 2/3)\, \phi_3\, dx = \phi_3(2/3) = 1$$

$$F = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

(e) (5 points) Solve the equation $KU = F$ and write down the FEM solution $U(x)$. Graph the solution.

$$\begin{pmatrix} 6 & -6 & 0 \\ -6 & 18 & -12 \\ 0 & -12 & 24 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Eq 1 $\quad 6U_1 = 6U_2 \Rightarrow \underline{U_1 = U_2}$

Eq 2 $\quad -6U_1 + 18U_2 - 12U_3 = 0 \rightarrow 12U_2 = 12U_3 \Rightarrow \underline{U_2 = U_3}$

Eq 3 $\quad -12U_2 + 24U_3 = 1 \quad 12U_3 = 1 \Rightarrow \underline{\underline{U_3 = \frac{1}{12}}}$

$$\Rightarrow \boxed{U_1 = U_2 = U_3 = \frac{1}{12}}$$

$$\boxed{U(x) = \frac{1}{12}\left(\phi_1(x) + \phi_2(x) + \phi_3(x)\right)}$$