

Exam 1

October 8, 2015

Time limit: 80 minutes

SOLUTIONS

Name (Print):

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	
	Total (%):	

This exam contains 10 pages (including this cover page) and 5 problems. Check to see if any pages are missing.

Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use your course textbook and notes. No electronic devices or other materials are allowed.

You are required to show your work on each problem on this exam, unless otherwise specified. The following rules apply:

- **Organize your work**, in a neat and coherent way. Work without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive very little credit; an incorrect answer supported by substantially correct calculations and explanations will still receive partial credit.
- **Box** your final answer to each problem.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem 1 (20 points):

(a) (6 points) Let's practice the LU factorization with an easy example. Compute the LU decomposition of the following matrix by hand. Show your steps.

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 7 \\ 1 & 4 & 10 \end{pmatrix} \quad (1)$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 7 \\ 1 & 4 & 10 \end{pmatrix} \xrightarrow{\substack{l_{21}=1 \\ l_{31}=1}} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 2 & 7 \end{pmatrix} \xrightarrow{l_{32}=1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{pmatrix}$$

U

$$\underline{\underline{L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}}}$$

(b) (2 points) Following from question (a), can we factorize A in LDL^T ? Why? If so, compute D .

No because A is not symmetric

(c) (2 points) Following from question (a), is A singular? Why?

No because $\det(A) = 1 \cdot 2 \cdot \cancel{3} = \underline{\underline{6}}$

(d) (2 points) Following from question (c), what can we say about the columns of A ?

They are linearly independent

(e) (2 points) What is the null space of A composed of?

The trivial solution $u = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(f) (6 points) Write a Matlab algorithm to calculate the LU factorization of a given $m \times m$ matrix A . Don't even think about writing $[L, U] = \text{lu}(A)$... Forget about trying to deal with special cases such as first pivot equal to zero, row reordering, etc. Suppose that A is as nice as in the questions above. (Hint #1: You can do it with only two *for* loops and two lines of commands within them. Hint #2: Initialize $U = A$ and $L = I$)

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U = A
L = eye(m)
for k = 1:m-1
    for j = k+1:m
        L(j,k) = U(j,k) / U(k,k)
        U(j,k:m) = U(j,k:m) - L(j,k) * U(k,k:m)
    end
end

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(c) (4 points) Suppose a new matrix B with eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$ and eigenvectors $y_1^T = 2^{-1/2}[-1 \ 1]$ and $y_2^T = 2^{-1/2}[-1 \ -1]$. Can you compute B without using B^{-1} ? Explain your answer and, if possible, compute B .

Note that $y_1^T \cdot y_2 = \frac{1}{\sqrt{2}}[-1 \ 1] \frac{1}{\sqrt{2}}[-1] = \frac{1}{2}(1-1) = 0 \Rightarrow y_1$ and y_2 are perpendicular

\Rightarrow 2 real eigenvalues
2 perpendicular eigenvectors $\Rightarrow B$ must be symmetric

$$\Rightarrow B = Q \Lambda Q^T = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -2 & -2 \end{bmatrix}$$

(d) (2 points) In the question above, what if y_1 and y_2 are not unit vectors? Can you still do the same?

Yes. Orthogonality is what really matters.

$$= \frac{1}{2} \begin{bmatrix} 3 & +1 \\ +1 & 3 \end{bmatrix}$$

B

(e) (4 points) Suppose

$$C = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

Can you diagonalize C in terms of its eigenvalues and eigenvectors? Explain your answer and, if possible, perform the diagonalization.

λ 's $\begin{vmatrix} (1-\lambda) & -1 \\ 1 & (-1-\lambda) \end{vmatrix} = (1-\lambda)(-1-\lambda) + 1 = -1 - \lambda + \lambda + \lambda^2 + 1 = 0$

$\Rightarrow \lambda_1 = \lambda_2 = 0$ (Repeated)
(i.e. C is singular)

$\Rightarrow (A - \lambda I)u = 0$

$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} y_a \\ y_b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow y = c \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow$ There is no second eigenvector
 \Rightarrow C cannot be diagonalized

(f) (2 points) For A , A^8 , B and C , determine if each matrix is positive definite, semidefinite or indefinite.

$A, A^8, B \rightarrow \lambda_1 > 0$ and $\lambda_2 > 0 \Rightarrow$ Positive definite

$C \rightarrow \det(C) = 0 \Rightarrow$ Positive semi-definite

Problem 3 (20 points): Consider the following boundary value problem:

$$-\frac{d^2u}{dx^2}(x) = f(x) \quad (2)$$

with $f(x) = -\delta_1(x - \frac{1}{4}) + \delta_3(x - \frac{3}{4})$ and fixed-fixed boundary conditions, i.e. $u(0)=u(1)=0$.
 (a) (8 points) Solve the problem analytically. Write the solution in the form:

$$u(x) = \begin{cases} \dots & \text{if } 0 < x < 1/4 \\ \dots & \dots \end{cases}$$

$$-\frac{d^2u}{dx^2} = -\delta_1(x - \frac{1}{4}) + \delta_3(x - \frac{3}{4})$$

$$u(0) = 0; \quad u(1) = 0$$

General solution

$$u(x) = R_1(x - \frac{1}{4}) - R_3(x - \frac{3}{4}) + C_1x + C_2$$

Boundary conditions

$$u(0) = 0 \rightarrow 0 = 0 - 0 + C_1 \cdot 0 + C_2 \rightarrow \underline{\underline{C_2 = 0}}$$

$$u(1) = 0 \rightarrow 0 = (x - \frac{1}{4}) - (x - \frac{3}{4}) + C_1$$

$$0 = (1 - \frac{1}{4}) - (1 - \frac{3}{4}) + C_1 \rightarrow \underline{\underline{C_1 = -\frac{3}{4} + \frac{1}{4} = -\frac{2}{4} = -\frac{1}{2}}}$$

Solution

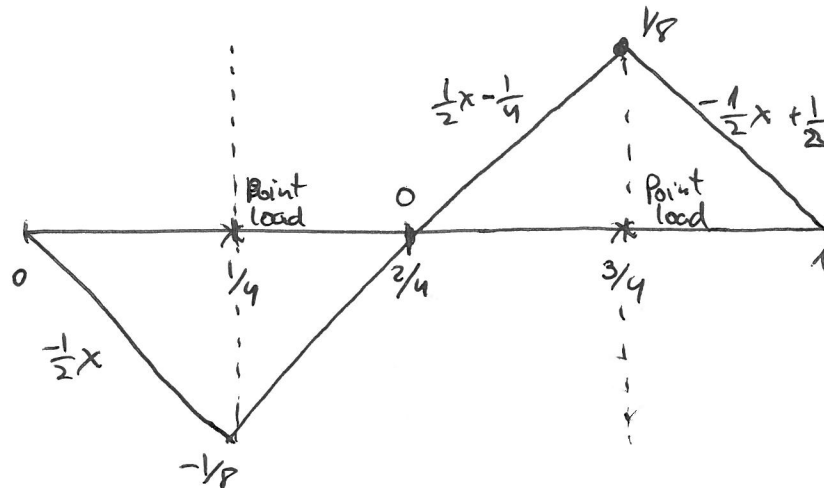
$$u(x) = R_1(x - \frac{1}{4}) - R_3(x - \frac{3}{4}) - \frac{1}{2}x$$

$$u(x) = \begin{cases} -\frac{1}{2}x & \text{if } 0 \leq x < \frac{1}{4} \\ \frac{1}{2}x - \frac{1}{4} & \text{if } \frac{1}{4} \leq x < \frac{3}{4} \\ -\frac{1}{2}x + \frac{1}{2} & \text{if } \frac{3}{4} \leq x \leq 1 \end{cases}$$

$$x = \frac{1}{4} \rightarrow u(x) = -\frac{1}{8}$$

$$x = \frac{3}{4} \rightarrow u(x) = \frac{1}{8}$$

(b) (6 points) Sketch the solution including the values of $u(x = 1/4)$ and $u(x = 3/4)$.



(c) (6 points) Setup but *do not solve* the discretized problem in matrix form $Au = b$ with a grid spacing of $h = 1/4$. The solution vector u to this linear system is our approximation to $u(x)$ at the grid points.

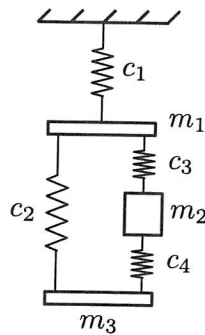
$$\begin{cases} -\Delta^2 u_i = -\delta_1 + \delta_3 & i=1, 2, 3 \\ u_0 = 0 ; u_4 = 0 \end{cases}$$

$$\frac{1}{h^2} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \frac{1}{h} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{h} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = h \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow u = (A \setminus b) h \Rightarrow u = \begin{bmatrix} -1/8 \\ 0 \\ 1/8 \end{bmatrix}$$

\uparrow
 This was not necessary

Problem 4 (20 points): For the following system of spring and masses,



(a) (3 points) What matrix A gives the spring stretching $e = Au$ from the displacements u of the masses?

$$\begin{cases} e_1 = u_1 \\ e_2 = u_3 - u_1 \\ e_3 = u_2 - u_1 \\ e_4 = u_3 - u_2 \end{cases} \rightarrow \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

\textcircled{A}

(b) (8 points) Find the stiffness matrix K of the system.

$$K = A^T C A$$

$$\begin{bmatrix} +1 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 & & & \\ & c_2 & & \\ & & c_3 & \\ & & & c_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

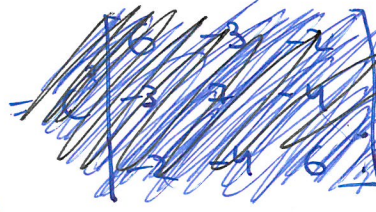
$$= \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 & 0 & 0 \\ -c_2 & 0 & c_2 \\ -c_3 & c_3 & 0 \\ 0 & -c_4 & c_4 \end{bmatrix} = \begin{bmatrix} (c_1 + c_2 + c_3) & -c_3 & -c_2 \\ -c_3 & (c_3 + c_4) & -c_4 \\ -c_2 & -c_4 & (c_2 + c_4) \end{bmatrix}$$

\textcircled{K}

(c) (2 points) Is there any zero entry in K ? Why?

No because all the masses are directly connected by springs

(d) (5 points) Suppose $c_1 = c$, $c_2 = 2c$, $c_3 = 3c$, $c_4 = 4c$ (with $c > 0$), demonstrate whether or not K is singular.

$$K = \begin{bmatrix} 6c & -3c & -2c \\ -3c & 7c & -4c \\ -2c & -4c & 6c \end{bmatrix}$$


$$\det(K) = c^3 \left[\frac{6 \cdot 7 \cdot 6}{252} + \frac{(-3)(-4)(-2)}{-24} + \frac{(-3)(-4)(-2)}{-24} - \frac{(-2)(7)(-2)}{28} - \frac{6(-3)(-3)}{54} - \frac{6(-4)^2}{96} \right]$$

$$\rightarrow \det(K) = c^3(252 - 226) = \underline{26c^3} > 0$$

K is invertible

(e) (2 points) Under the same conditions as in the previous question (d) demonstrate whether or not K is positive definite.

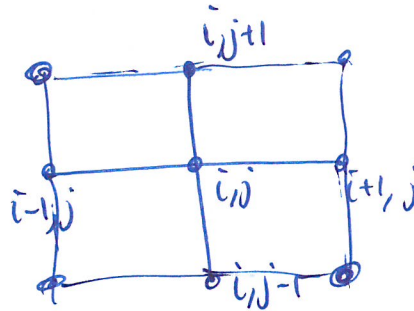
$$\det(K) > 0 \Rightarrow \underline{\text{Positive definite}}$$

Problem 5 (20 points): Approximation to mixed derivatives (2D). Suppose $u = u(x, y)$, find the finite approximation to

$$\left(\frac{\partial^2 u}{\partial x \partial y} \right). \quad (3)$$

Use always a centered scheme and show the order of accuracy.

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{d}{dx} \left(\frac{\partial u}{\partial y} \right)$$



$$\left(\frac{\partial^2 u}{\partial x \partial y} \right)_{i,j} = \frac{\left(\frac{\partial u}{\partial y} \right)_{i+1,j} - \left(\frac{\partial u}{\partial y} \right)_{i-1,j}}{2 \Delta x} + O(\Delta x^2)$$

$$\left(\frac{\partial u}{\partial y} \right)_{i+1,j} = \frac{u_{i+1,j+1} - u_{i+1,j-1}}{2 \Delta y} + O(\Delta y^2)$$

$$\left(\frac{\partial u}{\partial y} \right)_{i-1,j} = \frac{u_{i-1,j+1} - u_{i-1,j-1}}{2 \Delta y} + O(\Delta y^2)$$

$$\Rightarrow \left(\frac{\partial^2 u}{\partial x \partial y} \right)_{i,j} = \frac{u_{i+1,j+1} - u_{i+1,j-1} - u_{i-1,j+1} + u_{i-1,j-1}}{4 \Delta x \Delta y} + O(\Delta x^2, \Delta y^2)$$