18.085: Computational Science and Engineering I

Problem Set 8 Solutions

Fall 2014

4.1:

1) Find the Fourier series on $-\pi \le x \le \pi$ for ...

(a) $f(x) = \sin^3 x$, an odd function

$$\begin{aligned}
\sin(3x) &= \sin(x)\cos(2x) + \cos(x)\sin(2x) \\
&= \sin(x)\left[\cos^2(x) - \sin^2(x)\right] + 2\cos^2(x)\sin(x) \\
&= \sin(x) - 2\sin^3(x) + 2\sin(x) - 2\sin^3(x) \\
\sin(3x) &= 3\sin(x) - 4\sin^3(x)
\end{aligned}$$

$$\boxed{\sin^3(x) = \frac{3}{4}\sin(x) - \frac{1}{4}\sin(3x)}$$

(b) $f(x) = |\sin x|$, an even function

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\sin(x)| dx = \frac{1}{\pi}\int_0^{\pi} \sin(x)dx = \frac{1}{\pi}\left[-\cos(\pi) + \cos(0)\right] = \frac{2}{\pi}$$

First, the identity $2\sin(x)\cos(kx) = \sin((k+1)x) - \sin((k-1)x)$ for $k = 1, 2, \ldots$ is proven, which will be useful later.

$$\sin((k+1)x) - \sin((k-1)x) = \cancel{\sin(kx)\cos(x)} + \cos(kx)\sin(x) - \cancel{\sin(kx)\cos(x)} + \cos(kx)\sin(x) = 2\sin(x)\cos(kx)$$

For $k = 1, 2, \ldots$:

$$\begin{aligned}
a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin(x)|\cos(kx)dx = \frac{2}{\pi}\int_0^{\pi} \sin(x)\cos(kx)dx = \frac{1}{\pi}\int_0^{\pi} \left[\sin((k+1)x) - \sin((k-1)x)\right] \\
&= \frac{1}{\pi}\left[-\frac{1}{k+1}\cos((k+1)x) + \frac{1}{k-1}\cos((k-1)x)\right]_0^{\pi} \\
&= \frac{1}{\pi}\left[-\frac{1}{k+1}\cos((k+1)\pi) + \frac{1}{k-1}\cos((k-1)\pi) + \frac{1}{k+1}\cos(0) - \frac{1}{k-1}\cos(0)\right] \\
&= \begin{cases} 0, & \text{if } k \text{ is odd} \\ \frac{2}{\pi}\left[\frac{1}{k+1} - \frac{1}{k-1}\right], & \text{if } k \text{ is even} \end{cases} = \begin{cases} 0, & \text{if } k \text{ is odd} \\ -\frac{4}{\pi}\left[\frac{1}{k^2-1}\right], & \text{if } k \text{ is even} \end{cases}
\end{aligned}$$

$$\boxed{|\sin(x)| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{j=1}^{\infty} \frac{\cos((2j)x)}{(2j)^2 - 1}}$$

(c) $f(x) = x$, an odd function which can be treated as periodic over $-\pi$ to π

$$\begin{aligned}
b_k &= \frac{2}{\pi}\int_0^{\pi} x\sin(kx)dx = \frac{2}{\pi}\int_0^{\pi} x\frac{d}{dx}\left[-\frac{1}{k}\cos(kx)\right]dx = \frac{2}{k\pi}\int_0^{\pi}\frac{d}{dx}[x]\cos(kx)dx + \frac{2}{\pi}\left[-\frac{x}{k}\cos(kx)\right]_0^{\pi} \\
&= \frac{2}{k^2\pi}\left[\sin(k\pi) - \sin(0)\right] - \frac{2}{\cancel{\pi}}\left[\frac{\cancel{\pi}}{k}\cos(k\pi) + 0\right] = -\frac{2(-1)^k}{k}
\end{aligned}$$

$$\boxed{x = 2\sum_{k=1}^{\infty} \frac{(-1)^{k+1}\sin(kx)}{k} \quad \text{for } -\pi < x < \pi}$$

(d) $f(x) = e^x$, using the complex form of the series

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x e^{-ikx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{x(1-ik)}dx = \frac{1}{2\pi}\left[\frac{1}{1-ik}e^{x(1-ik)}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi(1-ik)}\left[e^{\pi}(\cos(k\pi) - \cancel{i\sin(k\pi)}) - e^{-\pi}(\cos(k\pi) + \cancel{i\sin(k\pi)})\right] = \frac{(-1)^k(e^{\pi} - e^{-\pi})}{2\pi(1-ik)}$$

$$\boxed{e^x = \frac{e^{\pi} - e^{-\pi}}{2\pi}\sum_{k=-\infty}^{\infty}\frac{(-1)^k e^{ikx}}{ik-1}}$$

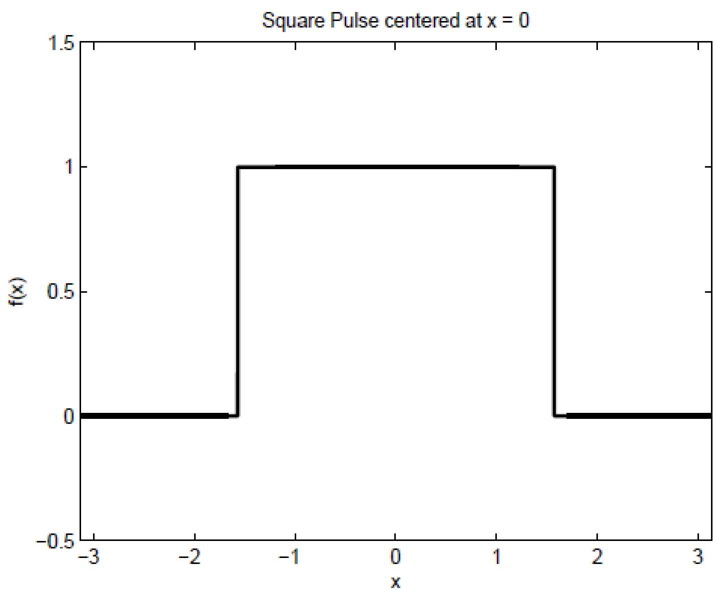
A function $F(x)$ can be split into even, $F_{\text{even}}(x)$, and odd, $F_{\text{odd}}(x)$, parts.

$$F(x) = F_{\text{even}}(x) + F_{\text{odd}}(x) \qquad F_{\text{even}}(x) = \frac{F(x) + F(-x)}{2} \qquad F_{\text{odd}}(x) = \frac{F(x) - F(-x)}{2}$$

The even an odd parts of e^x are $\cosh(x)$ and $\sinh(x)$, respectively.
The even and odd parts of e^{ix} are $\cos(x)$ and $i\sin(x)$, respectively.

3) A periodic square pulse centered at $x = 0$ is given by the following equation for $-\pi < x < \pi$:

$$f(x) = \begin{cases} 1, & \text{for } |x| < \frac{\pi}{2} \\ 0, & \text{for } \frac{\pi}{2} < |x| < \pi \end{cases}$$



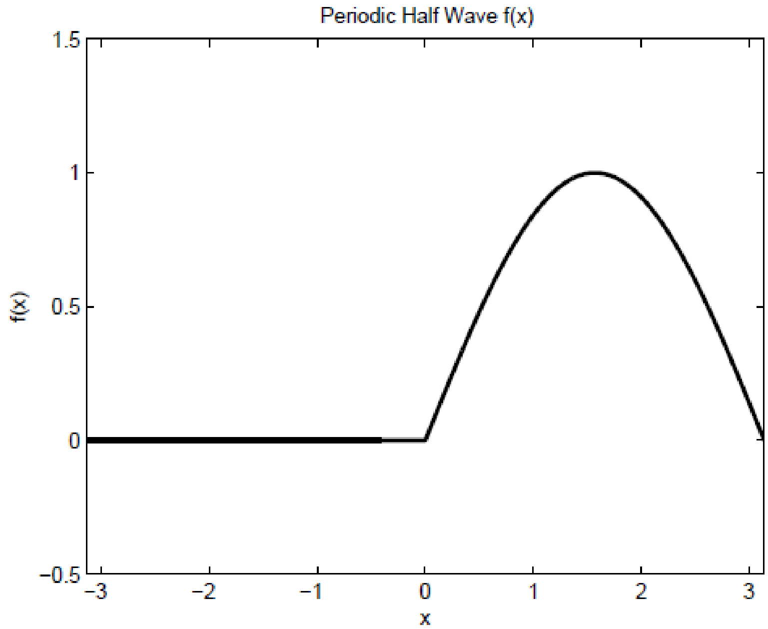
We would like to find the Fourier coefficients a_k, b_k for this square pulse. Because the pulse is an even function, $\boxed{b_k = 0}$.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx = \frac{1}{2\pi}\pi = \boxed{a_0 = \frac{1}{2}}$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos(kx)dx = \frac{1}{k\pi}\left[\sin\left(\frac{k\pi}{2}\right) - \sin\left(-\frac{k\pi}{2}\right)\right] = \boxed{a_k = \frac{2}{k\pi}\sin\left(\frac{k\pi}{2}\right)}$$

7)

$$f(x) = \begin{cases} \sin(x), & \text{for } 0 < x < \pi \\ 0, & \text{for } -\pi < x < 0 \end{cases}$$



The Fourier coefficient of $f(x)$ are:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_{0}^{\pi} \sin(x)dx = \frac{1}{2\pi}\left[-\cos(\pi)+\cos(0)\right] = \frac{1}{\pi}$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx = \frac{1}{\pi}\int_{0}^{\pi} \sin(x)\cos(kx)dx = \begin{cases} 0, & \text{if } k \text{ is odd} \\ -\frac{2}{\pi}\left[\frac{1}{k^2-1}\right] & \text{if } k \text{ is even} \end{cases} \quad \text{(See Problem 1b)}$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx = \frac{1}{\pi}\int_{0}^{\pi} \sin(x)\sin(kx)dx = \begin{cases} \frac{1}{2}, & \text{if } k = 1 \\ 0, & \text{if } k \neq 1 \end{cases}$$

$$\boxed{f(x) = \frac{1}{2}\sin(x) + \frac{1}{\pi} - \frac{2}{\pi}\sum_{j=1}^{\infty}\frac{\cos((2j)x)}{(2j)^2-1}}$$

Comparing the above Fourier series to that in problem 1b, it can be seen that the above equation is simply $f(x) = \frac{1}{2}\sin(x) + \frac{1}{2}|\sin(x)|$.

11)

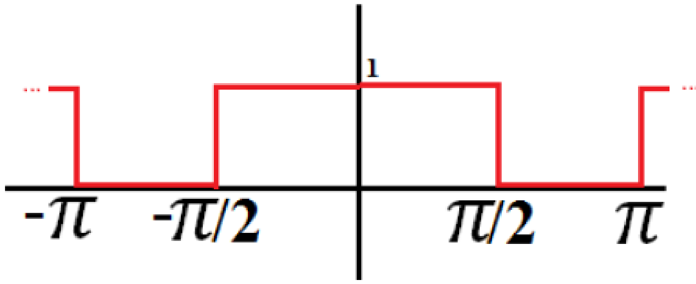
$$c_k = \frac{1}{2k}\ for\ k > 0, c_0 = 1$$

$$u_0 = 1 + \sum_{k=1}^{\infty} e^{ik\theta}\frac{1}{2k} = 1 - \frac{1}{2}\log(1-e^{i\theta})$$

$$u = 1 - \frac{1}{2}\log(1-re^{i\theta})$$

Section 4.1 (Problem 13):

For the following even square pulse $f(x) = \frac{1}{2} + \sum_{k=1}^{\infty} \operatorname{sinc}\left(\frac{\pi k}{2}\right) \cos kx$



(a) The energy $\int |f(x)|^2 dx$ is

$$\int_{-\pi}^{\pi} |f(x)|^2 \, dx = \int_{-\pi/2}^{\pi/2} 1 \, dx = \pi$$

(b) Note that

$$f(x) = \frac{1}{2} + \sum_{k=1}^{\infty} \operatorname{sinc}\left(\frac{\pi k}{2}\right) \cos kx = \frac{1}{2} + \sum_{k=1}^{\infty} \operatorname{sinc}\left(\frac{\pi k}{2}\right) \frac{e^{ikx} + e^{-ikx}}{2} \quad \Rightarrow \quad c_0 = \frac{1}{2} \text{ and } c_{k \neq 0} = \frac{1}{2} \operatorname{sinc}\left(\frac{\pi k}{2}\right)$$

(c) Using the energy identity, and noting that c_k is even, we get

$$\int_{-\pi}^{\pi} |f(x)|^2 \, dx = 2\pi \sum_{k=-\infty}^{\infty} |c_k|^2 = 2\pi \left(|c_0|^2 + 2 \sum_{k=1}^{\infty} |c_k|^2 \right) = 2\pi \left(\frac{1}{4} + \frac{1}{2} \sum_{k=1}^{\infty} \left| \operatorname{sinc}\left(\frac{\pi k}{2}\right) \right|^2 \right)$$

$$= 2\pi \left(\frac{1}{4} + \frac{2}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \right) = 2\pi \left(\frac{1}{4} + \frac{2}{\pi^2} \left(\frac{\pi^2}{8} \right) \right) = \pi$$

4.2:

4.2.1.a) We have

$$\begin{aligned} c_{mn} &= \left(\frac{1}{2\pi}\right)^2 \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} F(x,y)e^{-imx}e^{-iny}dxdy \\ &= \frac{1}{4\pi^2}\int_0^{\pi}\int_0^{\pi} e^{-imx}e^{-iny}dxdy = \frac{1}{4\pi^2}\int_0^{\pi} e^{-imx}dx\int_0^{\pi} e^{-iny}dy. \end{aligned}$$

Since $\int_0^{\pi} e^{-imx}dx$ equals π if $m = 0$, and $\frac{1-e^{-im\pi}}{im}$ otherwise, we get $c_{00} = 1/4, c_{m0} = c_{0m} = \begin{cases} 1/(2\pi im) & if\ m\ odd \\ 0 & otherwise \end{cases}$ for $m \neq 0$, and $c_{mn} = \begin{cases} \frac{-1}{\pi^2 mn} & if\ m,n\ odd \\ 0 & otherwise \end{cases}$ for $m,n \neq 0$.

b) $c_{mn} = \frac{1}{4\pi^2}\left(\int_0^{\pi}\int_0^{\pi} e^{-imx}e^{-iny}dxdy + \int_{-\pi}^{0}\int_{-\pi}^{0} e^{-imx}e^{-iny}dxdy\right)$. Similarly to a), we obtain $c_{00} = 1/2, c_{m0} = c_{0m} = 0$ for $m \neq 0$ and $c_{mn} = \begin{cases} \frac{-2}{\pi^2 mn} & if\ m,n\ odd \\ 0 & otherwise \end{cases}$ for $m,n \neq 0$.

4.2.2. Since $\sin mx$ and $\sin ny$ are odd in x and y respectively, $S(x,y)$ will have a double sine series $\sum\sum b_{mn}\sin mx \sin ny$ if $S(x,y)$ is odd in x and y: $-S(x,y) = S(-x,y) = S(x,-y)$. The double sine functions are orthogonal:

$$\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}(\sin kx \sin ly)(\sin mx \sin ny)dxdy = \begin{cases} 1 & if\ k = m\ and\ l = n \\ 0 & otherwise \end{cases}.$$

4.2: 3

3) A 2π-periodic, odd 2D function $F(x,y)$ has a double sine series:

$$F(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{nm}\sin(nx)\sin(my)$$

The coefficients b_{nm} can be found through the formula

$$\boxed{b_{nm} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} F(x,y)\sin(nx)\sin(ny)dxdy}$$

4.3:

2) The jth row (where $j = 0$ is the first row and $j = N-1$ is the last row) of the Fourier matrix F_N is a row vector

$$\begin{bmatrix} 1 & w^j & w^{2j} & \dots & w^{(N-1)j} \end{bmatrix} \tag{1}$$

where $w = \exp(i2\pi/N)$. The $(N-j)$th row vector of F_N is

$$\begin{bmatrix} 1 & w^{(N-j)} & w^{2(N-j)} & \dots & w^{(N-1)(N-j)} \end{bmatrix}$$

which can be also written as

$$\begin{bmatrix} 1 & \overline{w^j} & \overline{w^{2j}} & \dots & \overline{w^{(N-1)j}} \end{bmatrix} \tag{2}$$

by noting that the complex conjugate of w^{kj} is $\overline{w^{kj}} = w^{k(N-j)}$. The row vector 2 is simply the complex conjugate of the row vector 1, or alternatively, the $(N-j)$th row of F_N is identical to the jth row of the $\overline{F}_N$. The proof of $\overline{w^{kj}} = w^{k(N-j)}$ is given below:

$$\overline{w^{kj}} = \left(\overline{w^k}\right)^j = \left(\overline{\exp\left(\frac{i2\pi k}{N}\right)}\right)^j = 1^k \left(\exp\left(\frac{-i2\pi k}{N}\right)\right)^j = \left(\exp\left(\frac{i2\pi N}{N}\right)\right)^k \left(\exp\left(\frac{-i2\pi k}{N}\right)\right)^j$$

$$= \exp\left(\frac{i2\pi kN}{N}\right)\exp\left(-\frac{i2\pi kj}{N}\right) = \exp\left(\frac{i2\pi k(N-j)}{N}\right) = \left[\exp\left(\frac{i2\pi}{N}\right)\right]^{k(N-j)} = w^{k(N-j)}$$

8) $c = (1,0,1,0)$, $\quad N = 4$

$c' = (1,1) \qquad c'' = (0,0)$

$f' = F_2 c' \qquad f'' = F_2 c''$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0 \end{bmatrix} \qquad = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

First half,

$$f_j = f'_j + (w_N)^j f''_j$$

$$f_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & \\ & i \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Second half,

$$f_{j+M} = f'_j - (w_N)^j f''_j$$

$$f_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & \\ & i \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\therefore \text{Combine } f = [f_1\,;\, f_2] = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix} \#$$

For $c = (0,1,0,1)$

$c' = (0,0) \qquad c'' = (1,1)$

$f' = F_2 c' \qquad f'' = F_2 c''$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

10) If $w = \exp\left(\frac{2\pi i}{64}\right)$ then w^2 and $\sqrt{w}$ are among the $\underline{32^{\text{th}}}$ and $\underline{128^{\text{th}}}$ roots of 1, respectively.