Solutions to Homework 11 (Section 4.5)

(1) Changing to $-e^{ax}$ for $x<0$ leads to

$$\hat{g}(k) = \frac{1}{a+ik} - \frac{1}{a-ik} = \frac{(a-ik)-(a+ik)}{(a+ik)(a-ik)} =$$

$$= \frac{-2ik}{a^2+k^2}$$ This is $O\left(\frac{1}{k}\right)$ from the jump at 0

(2) (a) $f=1$ on $[0,L]$ $\displaystyle\int_0^L e^{-ikx}\,dx = \frac{1-e^{-ikL}}{ik}$

(b) The limit $a \to 0$ in Prob. 1 gives $-\dfrac{2ik}{k^2} = \dfrac{-2i}{k}$

(c) $\hat{f}(k)$ must be $\begin{cases} 2\pi & 0<k<1 \\ 0 & \text{else} \end{cases}$ since $\displaystyle\int_0^1 e^{ikx}\,dk$ reproduces $f(x)$

(d) $f(x) = \sin x = (e^{ix} - e^{-ix})/2i$

so $\hat{f}(k) = \displaystyle\int_0^{4\pi} \left(\frac{e^{ix}-e^{-ix}}{2i}\right) e^{-ikx}\,dx =$ STANDARD INTEGRALS

(8) $\displaystyle\int f(ax)\,e^{-ikx}\,dx = \int f(y)\,e^{-i\frac{k}{a}y}\,\frac{dy}{a} = \frac{1}{a}\hat{f}\left(\frac{k}{a}\right)$

set $y = ax$

(11) $\dfrac{du}{dx} + au = \delta(x-d) \to (ik+a)\,\hat{u}(k) = e^{-ikd}$

SHIFT RULE

Then $\hat{u}(k) = \dfrac{1}{a+ik}e^{-ikd}$ = transform of shifted 2-way pulse $u(x)$

(12) $\dfrac{1}{ik}\hat{u}(k) - ik\,\hat{u}(k) = 1 \longrightarrow \hat{u}(k) = \dfrac{1}{\frac{1}{ik} - ik} = \dfrac{ik}{1+k^2}$

$u(x) =$ derivative of $\frac{1}{2}$(2-way pulse with $a=1$)
(from ik)

(22) $\left((ik)^4 - 2(ik)^2 + 1\right)\hat{G}(k) = 1$

$$\hat{G}(k) = \frac{1}{k^4 + 2k^2 + 1} = \frac{1}{(k^2+1)^2}$$

Since $f(x) = \frac{1}{2}e^{-|x|} =$ 2-way pulse with $a = 1$ has $\hat{f}(k) = \frac{1}{k^2+1}$

we can ~~a~~ compute $G(x)$ by convolution $f(x) * f(x)$

(23) $\delta * \delta = \delta$

(24) $\int_{-\infty}^{0} e^{5x} e^{-ikx}\,dx + \int_{0}^{\infty} e^{-3x} e^{-ikx}\,dx =$ standard integrations

If $\hat{f}(k) = e^{-|k|}$ then $f(x) = \frac{C}{1+x^2}$
(2-way pulse)
(just reversing f and $\hat{f}$, and 2π will enter)

(25) $\hat{f}(k_1, k_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(k_1x + k_2y)} f(x,y)\,dx\,dy$

Then $f(x,y) = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{f}(k_1,k_2)\,e^{i(k_1x+k_2y)}\,dk_1\,dk_2$