18085 FALL 2001 QUIZ 3 SOLUTIONS

PROBLEM 1

(a)

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-ikx} dx$$
$$\hat{f}(k) = \int_{-\pi}^{\pi} f(x)\, e^{-ikx} dx$$

They are equal to within a factor of 2π.

(b)

The integral recovers

$$\phi(x) = \begin{cases} f(x); & -\pi < x < \pi \\ 0; & \text{otherwise} \end{cases}$$

The sum recovers

$$\phi(x) = f(x \bmod 2\pi)$$

So if we think of this in terms of "information carried by $f(x)$" we see that the discrete sum recovers all of it. This is quite surprising. The reason why this is possible (and you'll learn this if you ever take signal processing or wavelets) is that $f(x)$ is "band-limited" - it is only non-zero on $[-\pi; \pi]$. If, say, $f(x)$ extended beyond this interval then for one thing it is not clear what it means to extend it periodically. But even if we defined it somehow (add on overlaps?) then (discrete) recovery process will introduce distortion known as aliasing.

To make you feel better, I had no idea about this question until Prof. Strang reminded me of what I learned in his wavelet course.

(c)

The hat function is continuous, so

$$c_k^{hat} \sim \frac{1}{k^2}$$

The square of the hat function is also merely continuous (the derivative is discontinuous at $x = 0$) and so

$$c_k^{hat^2} \sim \frac{1}{k^2}$$

PDF created with FinePrint pdfFactory trial version http://www.fineprint.com

PROBLEM 2

(a)

We have

$$H(k) = F(k)B(k)$$

where

$$B(k) = \int_{-1/2}^{1/2} e^{-ikx}dx = -\frac{e^{-ik/2} - e^{ik/2}}{ik} = \frac{\sin\frac{k}{2}}{\frac{k}{2}}$$

and so

$$H(k) = F(k)\frac{\sin\frac{k}{2}}{\frac{k}{2}}$$

As we discussed on Thursday, $h(x)$ is more smooth than $f(x)$ by "1 unit". This is because in the Fourier space $F(k)$ is multiplied by $O\left(\frac{1}{k}\right)$

(b)

In the Fourier domain, differentiation becomes multiplication by $-ik$. So

$$\hat{h}'(k) = 2iF(k)\sin\frac{k}{2}$$

(c)

We bring the differentiation sign inside the integral.

$$\begin{aligned} h'(x) &= \frac{d}{dx}\int_{-\frac{1}{2}}^{\frac{1}{2}} f(x-y)\,dy = \int_{-\frac{1}{2}}^{\frac{1}{2}} f'(x-y)\,dy = \\ &= f(x-y)|_{-\frac{1}{2}}^{\frac{1}{2}} = f\left(x+\frac{1}{2}\right) - f\left(x-\frac{1}{2}\right) \end{aligned}$$

We used the Fundamental Theory of Calculus. For further clarification see the write-up on the Chain rule.

Now (sub $z = x - \frac{1}{2}$ and $t = x + \frac{1}{2}$)

$$\begin{aligned} \hat{h}'(k) &= \int_{-\infty}^{\infty} f\left(x+\frac{1}{2}\right)e^{-ikx}dx - \int_{-\infty}^{\infty} f\left(x-\frac{1}{2}\right)e^{-ikx}dx = \\ &= \int_{-\infty}^{\infty} f(z)\,e^{-ik\left(z-\frac{1}{2}\right)}dz - \int_{-\infty}^{\infty} f(f)\,e^{-ik\left(t+\frac{1}{2}\right)}dt = \\ &= e^{ik\frac{1}{2}}F(k) - e^{-ik\frac{1}{2}}F(k) = 2iF(k)\sin\frac{k}{2} \end{aligned}$$

PDF created with FinePrint pdfFactory trial version http://www.fineprint.com

PROBLEM 3

(a)

$$\omega_8 = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$
$$\omega_4 = i$$

...and we also know that

$$\omega_4 = \omega_8^2$$

(b)

The correct statement of the problem has

$$\begin{aligned} Y_1 &= c_0 + c_2\omega_4 + c_4\omega_4^2 + c_6\omega_4^3 \\ Z_1 &= c_1 + c_3\omega_4 + c_4\omega_5^2 + c_7\omega_4^3 \\ y_2 &= c_0 + c_1\omega_8 + c_2\omega_8^2 + c_3\omega_8^3 + c_4\omega_8^4 + c_5\omega_8^5 + c_6\omega_8^6 + c_7\omega_8^7 \end{aligned}$$

and so

$$\begin{aligned} y_2 &= c_0 + c_2\omega_8^2 + c_4\omega_8^4 + c_6\omega_8^6 + \omega_8\left(c_1\omega_8 + c_3\omega_8^2 + c_5\omega_8^4 + c_7\omega_8^6\right) = \\ &= Y_1 + \omega_8 Z_1 \end{aligned}$$

(c)

h is a 4-tap averaging filter. So

$$h(n) = \left[\ldots; 0; 0; \frac{1}{4}; \frac{1}{4}; \frac{1}{4}; \frac{1}{4}; 0; 0; \ldots\right]$$

And this is the vector which makes the expression for $y(n)$ look like a convolution

$$y(n) = \frac{1}{4}\sum_{k=0}^{3} x(n-k) = \sum_{k=-\infty}^{\infty} h(k)\,x(n-k)$$

(d)

We want to obtain the fact that the transform of convolutions is the product of transforms. We have

$$\begin{aligned} Y(\omega) &= \sum_{n=-\infty}^{\infty} y(n)e^{-in\omega} = \sum_{n=-\infty}^{\infty}\sum_{k-\infty}^{\infty} h(k)\,x(n-k)\,e^{-in\omega} = \text{(why?)} \\ &= X(\omega)\sum_{k-\infty}^{\infty} h(k)e^{-ik\omega} = X(\omega)H(\omega) \end{aligned}$$

PDF created with FinePrint pdfFactory trial version http://www.fineprint.com

In our case, we have

$$Y(\omega) = \frac{1}{4} X(\omega)\left(1 + e^{-i\omega} + e^{-2i\omega} + e^{-3i\omega}\right)$$

Since H is a 4-tap averaging filter, any signal such that the sum of any 4 consecutive elements is 0 satisfies the problem. For example,

$$[\ldots, 4, -2, -2, 0, 4, -2, -2, 0, \ldots]$$

PDF created with FinePrint pdfFactory trial version http://www.fineprint.com