Prob 1.

a) For a complex function \( f(z) = u(x,y) + i\, v(x,y) \) to be an analytic function, its imaginary part \( v(x,y) \) must satisfy the Laplace eqn:

\[ v_{xx} + v_{yy} = 0 \]

Let \( v(x,y) = xe^{-y} \)

\[
\begin{align*}
v_x &= e^{-y} \\
v_y &= xe^{-y} \\
v_{xx} &= 0 \\
v_{yy} &= xe^{-y}
\end{align*}
\]

\( v_{xx} + v_{yy} = 0 \) is not true for all \( x, y \in \mathbb{R} \). Therefore there is no analytic function the imaginary part of which is \( xe^{-y} \).

b) If \( u = x^4 + ax^2y^2 + y^4 \) is the real part of an analytic function, \( u \) must satisfy the Laplace eqn:

\[ u_{xx} + u_{yy} = 0 \]

\[
\begin{align*}
u_x &= 4x^3 + 2ay^2x \\
u_y &= 2ax^2y + 4y^3 \\
u_{xx} &= 12x^2 + 2ay^2 \\
u_{yy} &= 2ax^2 + 12y^2
\end{align*}
\]

For \( u_{xx} + u_{yy} = 12x^2 + 2ay^2 + 2ax^2 + 12y^2 = 0 \) for all \( x, y \in \mathbb{R} \), \( a = -6 \).

\[ a = -6 \]

2. An analytic function satisfies the Cauchy–Riemann equations:

\[
\begin{align*}
u_x &= v_y, \\
u_y &= -v_x
\end{align*}
\]

\[
\begin{align*}
v_y &= 4x^3 - 12y^2 \\
v_x &= 12x^2y - 4y^3
\end{align*}
\]

\[ v(x,y) = \int (4x^3 - 12y^2) \, dy, \quad v(x,y) = \int 12x^2y - 4y^3 \, dx 
\]

\[ v(x,y) = 4x^3y - 4y^3x + C \]

where \( C \in \mathbb{R} \).

3. \( f(z) = u(x,y) + i\, v(x,y) \)

\[
\begin{align*}
f(z) &= x^4 - 6x^2y^2 + y^4 + i\left(4x^3y - 4y^3x + C\right) \\
&= x^4 + 4x^3iy - 6x^2y^2 - 4iy^3x + y^4 + iC \\
&= (x+iy)^4 + iC \\
&= z^4 + iC
\end{align*}
\]

\[ z^4 + iC \]
Prob 2
Let \( u \) satisfy the two-dimensional Laplace eqn \( u_{xx} + u_{yy} = 0 \) in a region \( R \) where \( R \) is the disk \( x^2 + y^2 < 9 \) and \( u \) satisfies the boundary condition \( u = \cos^2 4\theta \) at \( r = 3 \). Find \( u \).

We can first rewrite the boundary condition as

\[
 u(3, \theta) = \cos^2 4\theta = \frac{\cos(8\theta) + 1}{2} \quad \checkmark
\]

For a \( u \) that satisfies the 2D Laplace eqn, we must look for a function which is analytical inside the disk of radius 3.

\[
 u(3, \theta) = \cos^2 4\theta
\]

We know that \( f(z) = z^n \) is entire, so we're looking for a \( u \) of the form \( r^n \cos(\theta n) \) that satisfies the boundary condition

\[
 \frac{\cos(8\theta) + 1}{2} = k \frac{3^n \cos(\theta n) + C}{2}
\]

Therefore \( k = \frac{1}{2 \cdot 3^8} \), \( n = 8 \), \( C = \frac{1}{2} \)

\[
 u(r, \theta) = \frac{1}{2} \cdot \frac{1}{3^8} \cdot r^8 \cos(8\theta) + \frac{1}{2} \quad \checkmark
\]

Prob 3
2) Prove the Liouville thm that if an entire function \( f(z) \) is bounded at infinity, it is a constant.

The Cauchy Integral formula says that if \( f(z) \) is analytic in the region \( R \) and if \( z \) is an interior point of \( R \)

\[
 f(z) = \frac{1}{2\pi i} \oint_C \frac{f(z')}{z'-z} \, dz',
\]

Differentiate w/ respect to \( z \)

\[
 f'(z) = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z'-z)^2} \, dz', \quad \checkmark
\]

Choose \( C \) to be circle \( C_R \), with center \( z_0 \) and radius \( R \)

\[
 |f'(z_0)| = \left| \frac{1}{2\pi i} \oint_{C_R} \frac{f(z)}{(z-z_0)^2} \, dz \right|
 \]

\[
 = \frac{1}{2\pi} \int_{C_R} \left| \frac{1}{z-z_0} \right| \, dz
\]
A point \( z \) on \( CR \) satisfies \( \mid z - z_0 \mid = R \)

\[
1{\mid f'(z_0) \mid} = \frac{1}{2\pi R} \int_{CR} \frac{\mid f(z) \mid}{R^2} \, d\bar{z}
\]

Because \( f(z) \) is bounded, we know \( \mid f(z) \mid \) on \( CR \) will be finite, (less than some constant \( K \) )

\[
1{\mid f'(z_0) \mid} \leq \frac{K}{2\pi R^2} \int_{CR} \, d\bar{z}
\]

Contour integral over circle \( w/ \) radius \( R \to circumference = 2\pi R \)

\[
1{\mid f(z_0) \mid} \leq \frac{K}{R} \quad \checkmark
\]

\[\text{lim } R \to \infty, \quad f'(z_0) = 0\]

Therefore, if \( f(z) \) is bounded at infinity, it is a constant.

b. Prove that if an entire function divided by \( z^n \) is bounded at infinity, it is a polynomial of order no more than \( n \).

Cauchy integral formula

\[
f(z) = \frac{1}{2\pi i} \int_{C} \frac{f(z')}{z' - z} \, dz'
\]

Differentiating \( m \) times:

\[
f^{(m)}(z) = \frac{m!}{2\pi i} \int_{C} \frac{f(z')}{(z' - z)^{m+1}} \, dz'
\]

Entire function divided by \( z^n \) bounded at infinity means:

\[
\frac{1{\mid f(z) \mid}}{z^n} \leq K \quad \rightarrow \quad 1{\mid f(z) \mid} \leq K \mid z^n \mid
\]

Take result generalized from before \((*)\)

\[
f^{(m)}(z) = \frac{1}{2\pi i R^{m+1}} \int_{CR} 1{\mid f(z) \mid} \, dz
\]

\[
= \frac{K \mid z^n \mid}{2\pi i R^{m+1}} \int_{CR} \, dz
\]

\[
f^{(m)}(z) = \frac{K \mid z^n \mid}{R^m}
\]

as \( R \to \infty \), derivatives will disappear for all \( m \geq n \) so

\( f \) must be a polynomial of order no more than \( n \).

\( \checkmark \)