Suggested problems

Frobenius • $x(1-x)y'' - 2y' + 2y = 0$; sol. around $x=0$

$$\longrightarrow c_0 s(s-3)x^{s-1} + \sum_{k=0}^{\infty}\Big[c_{k+1}(k+s+1)(k+s-2) - c_k(k+s+1)(k+s-2)\Big]x^{k+s} = 0$$

$s=0 \Rightarrow c_0$ arbitrary

$(c_{k+1} - c_k)(k-2)(k+1) = 0$; $k \geq 0$

$k=0 \Rightarrow c_1 = c_0$

$k=1 \Rightarrow c_2 = c_1$

$k=2 \Rightarrow c_3$ arbitrary

$k \geq 3 \Rightarrow c_{k+1} = c_k$

$$\Rightarrow y = c_0(1+x+x^2) + c_3 x^3(1+x+x^2+\cdots)$$

$$= c_0(1+x+x^2) + c_3\frac{x^3}{1-x} = \frac{c_1 + c_2 x^3}{1-x}$$

• $xy'' - xy' - y = 0$; sol. around $x=0$

$$\longrightarrow c_0 s(s-1)x^{s-1} + \sum_{k=0}^{\infty}\Big[c_{k+1}(k+s+1)(k+s) - c_k(k+s+1)\Big]x^{k+s} = 0$$

$s=1 \Rightarrow c_0$ arbitrary $\Rightarrow (k+2)\left[c_{k+1}(k+1) - c_k\right] = 0$; $k \geq 0$

$$\Rightarrow c_{k+1} = \frac{c_k}{k+1} = \frac{c_0}{(k+1)!} \Rightarrow y_1 = xe^x$$

$s=0 \longrightarrow$ same sol.

$\Rightarrow y_2 = Cy_1 \log x + v(x)$

$\underset{\text{ODE}}{\Longrightarrow}$ $v''x - v'x - v = -C(1+x)e^x$

$$= -C\sum_{k=0}^{\infty}\frac{x^k}{k!} - C\sum_{k=1}^{\infty}\frac{x^k}{(k-1)!}$$

$\Rightarrow v = \sum_{k=0}^{\infty} c_k x^k \Rightarrow c_{k+1}(k+1)k - c_k(k+1) = -\frac{C}{k!} \left(-\frac{C}{(k-1)!}\right.$ if $k \geq 1$

$k=0 \Rightarrow c_0 = -C$; c_1 arbitrary

$k=1 \Rightarrow c_2 = -C + c_1$

$k=2 \Rightarrow c_3 = -\frac{3C}{4} + c_1$ ...

$$\Rightarrow v = -C + c_1 x + (c_1 - C)x^2 + \left(c_1 - \frac{3C}{4}\right)x^3 \ldots$$

Sturm - Liouville

- Show that the eigenfunctions satisfying

$$y'' + \mu^2 y = 0 \quad ; \quad y(0) = 0 \quad ; \quad \alpha L y'(L) + y(L) = 0$$

verify $\int_0^L \sin(\mu_n x)\sin(\mu_m x)\,dx$

where $\tan(\mu_n L) + \alpha \mu_n L = 0$

$y = A\cos(\mu x) + B\sin(\mu x)$

$y(0) = 0 \Rightarrow A = 0$

$y'(L) = B\mu\cos(\mu L) \Rightarrow \alpha L y'(L) + y(L)$

$= B \cdot [\alpha L \mu \cos(\mu L) + \sin(\mu L)] = 0$

$\Rightarrow \tan(\mu L) + \alpha L \mu = 0$

- Fourier series solution of $y'' + \Lambda y = f(x)$

with $y(-\pi) = y(\pi)$; $y'(-\pi) = y'(\pi)$; $f(x) = \begin{cases} 0; & x \in (-\pi, 0) \\ 1; & x \in (0, \pi/2) \\ 0; & x \in (\pi/2, \pi) \end{cases}$

for $\Lambda \neq p^2$; $p \in \mathbb{N}$

$\rightarrow$ Fourier series $y = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi) + b_n \sin(n\pi)$

$$\Rightarrow a_0 \Lambda + \sum_{n=1}^{\infty} (\Lambda - n^2)(a_n \cos n\pi + b_n \sin n\pi) = f(x)$$

$$\int_{-\pi}^{\pi} a_0 \Lambda \, dx = \int_{-\pi}^{\pi} f(x)\,dx = \frac{\pi}{2} \Rightarrow a_0 = \frac{1}{4\Lambda}$$

$$\int_{-\pi}^{\pi} (LHS)\cdot \cos mx \, dx = (\Lambda - m^2)\,\pi a_m = \int_{-\pi}^{\pi} f(x)\cos(mx)\,dx = \frac{\sin(m\pi/2)}{m}$$

$$\Rightarrow a_n = \frac{\sin(n\pi/2)}{\pi n(\Lambda - n^2)}$$

$$\int_{-\pi}^{\pi} (LHS)\cdot \sin mx\,dx = (\Lambda - m^2)\,\pi b_m = \int_{-\pi}^{\pi} f(x)\sin(mx)\,dx = \frac{1-\cos m\pi/2}{m}$$

$$\Rightarrow b_n = \frac{1-\cos n\pi/2}{\pi n(\Lambda - n^2)}$$

$$\Rightarrow y = \frac{1}{4\Lambda} + \sum_{n=1}^{\infty} \frac{\sin(n\pi/2)\cos(nx) + [1-\cos n\pi/2]\sin(nx)}{\pi n(\Lambda - n^2)}$$

- Expand $f(x) = x^p$ in a series of eigenfunctions of the boundary-value problem

$$x\frac{d}{dx}(xy') + (\mu^2 x^2 - p^2)y = 0 \; ; \; y(0) \text{ finite}, \; y'(L) = 0$$

over the interval $0 < x < L$, where $p \in \mathbb{R}^+$

$\rightarrow$ Bessel eq. $\Rightarrow y = A\, J_p(\mu x) + B\, Y_p(\mu x)$

$y(0)$ finite $\Rightarrow B = 0$

$y'(L) = 0 \Rightarrow J_p'(\mu_n L) = 0 \qquad y_n = J_p(\mu_n x)$

$$x^p = \sum_{n=1}^{\infty} c_n J_p(\mu_n x)$$

$$c_n = \frac{\int_0^L x^{p+1} J_p(\mu_n x)\,dx}{\int_0^L x [J_p(\mu_n x)]^2\,dx}$$

$$\int_0^L x^{p+1} J_p(\mu_n x)\,dx = \frac{1}{\mu_n}\left[x^{p+1} J_{p+1}(\mu_n x)\right]_0^L$$

$$= \frac{L^{p+1}}{\mu_n} J_{p+1}(\mu_n L)$$

$$\int_0^L x\left[J_p(\mu_n x)\right]^2 dx = \frac{1}{2\mu_n^2}\left[\left[x J_p'(\mu_n x)\right]^2 + \left(\mu_n^2 x^2 - p^2\right) J_p^2(\mu_n x)\right]_0^L$$

$$= \frac{\left(\mu_n^2 L^2 - p^2\right)\left[J_p(\mu_n L)\right]^2}{2\mu_n^2} + \frac{p^2}{2\mu_n^2}\left[J_p(0)\right]^2$$

Math world $\Rightarrow \left[x^{-p} J_p(x)\right]' = -x^{-p} J_{p+1}(x)$

$$\Rightarrow \frac{d}{dx}\left[x^{-p} J_p(\mu_n x)\right] = -\mu_n x^{-p} J_{p+1}(\mu_n x)$$

$$= -p x^{-p-1} J_p(\mu_n x) + x^{-p}\mu_n J_p'(\mu_n x)$$

$$\xRightarrow{J_p'(\mu_n L) = 0} \quad J_{p+1}(\mu_n L) = \frac{p}{\mu_n L} J_p(\mu_n L) \quad ; p \neq 0$$

$$\Rightarrow c_n = \frac{\dfrac{2L^{p+1}\mu_n}{\cancel{\mu_n}} \cdot \dfrac{p}{\mu_n L} J_p(\mu_n L)}{\left(\mu_n^2 L^2 - p^2\right)\left[J_p(\mu_n L)\right]^2 + p^2\left[J_p(0)\right]^2}$$

$$p = 0 \Rightarrow c_n = 0 \Rightarrow 1 = 1$$

$$p \neq 0 \xRightarrow{J_p(0) = 0} c_n = \frac{2L^p \cdot p}{\left(\mu_n^2 L^2 - p^2\right) J_p(\mu_n L)}$$

- Find the three first coefs. in the expansion of $f(x) = \begin{cases} 0, & x \in (-1,0) \\ x, & x \in (0,1) \end{cases}$ in a series of Legendre polynomials over $(-1,1)$

$$\Rightarrow c_n = \frac{\int_{-1}^{1} f(x) P_n(x)\,dx}{\int_{-1}^{1} [P_n(x)]^2 dx} = \frac{2n+1}{2} \int_0^1 x\, P_n(x)\,dx$$

$$\Rightarrow c_0 = \frac{1}{2} \int_0^1 x\,dx = \frac{1}{4} \;;\; c_1 = \frac{3}{2} \int_0^1 x^2 dx = \frac{1}{2}$$

$$c_2 = \frac{5}{2} \int_0^1 x \cdot \frac{1}{2}(3x^2-1)\,dx = \frac{5}{16}$$

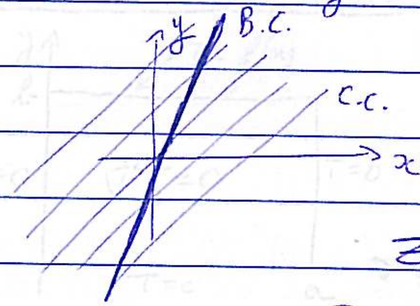
$$\Rightarrow f(x) \simeq \frac{P_0(x)}{4} + \frac{P_1(x)}{2} + \frac{5P_2(x)}{16} + \cdots$$

Suggested problems

Find, if possible, the particular solution to the following PDEs, that satisfies the B.C. $z(x, y=2x) = x^2$

1) $\partial_x z + \partial_y z + 2xz = 0$

$$\longrightarrow dx = dy = -\frac{dz}{2xz} \Rightarrow \begin{cases} u_1 = y - x \\ u_2 = \ln z + x^2 \end{cases}$$



$$\Rightarrow z = e^{-x^2} F(y-x)$$

$$z(x, y=2x) = F(x) e^{-x^2} = x^2 \Rightarrow F(x) = x^2 e^{x^2}$$

$$\Rightarrow \boxed{z = (y-x)^2 e^{(y^2-2xy)}}$$

2) $x^2 \partial_x z + y^2 \partial_y z = z^2$

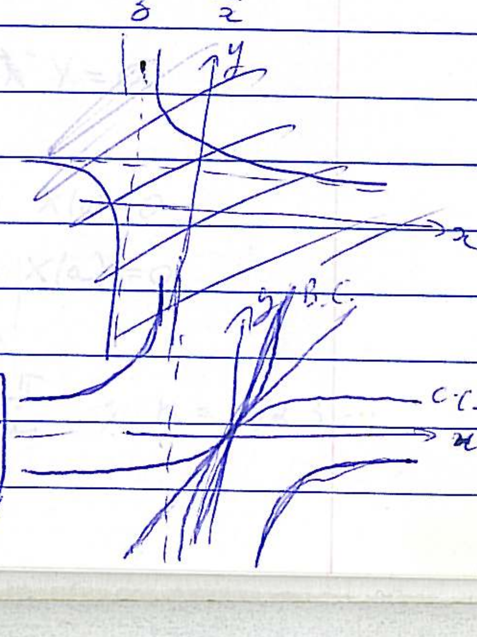
$$\longrightarrow \frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z^2} \Rightarrow \begin{cases} u_1 = \frac{1}{x} - \frac{1}{y} \\ u_2 = \frac{1}{z} - \frac{1}{x} \end{cases}$$

$$\Rightarrow z = \frac{1}{F\left(\frac{1}{x} - \frac{1}{y}\right) + \frac{1}{x}}$$

$$z(x, y=2x) = \frac{1}{F\left(\frac{1}{2x}\right) + \frac{1}{x}} = x^2$$

$$\Rightarrow F(x) = 4x^2 - 2x$$

$$\Rightarrow \boxed{z = \frac{x^2 y^2}{4x^2 - 8xy + 4y^2 - xy^2 + 2x^2 y}}$$



3) $x z \partial_x z + y z \partial_y z = xy$

$$\rightarrow \frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy} \Rightarrow \begin{cases} \frac{dx}{x} = \frac{dy}{y} \rightarrow u_1 = y/x \\ 2z\,dz = y\,dx + x\,dy \\ \quad = d(xy) \\ \quad \Rightarrow u_2 = z^2 - xy \end{cases}$$

$$\Rightarrow z = \sqrt{xy + F(y/x)}$$

$y = 2x$ is a C.C. $\Rightarrow$ impossible to impose the B.C.