18.075 - Solution set 4: About solving PDEs to preserve chocolate and make nice music

1 The insulating properties of a cylinder (60 pts)

With θ, the differential equation becomes

$$\partial_{\tilde{t}}\theta - \partial_{\tilde{r}\tilde{r}}\theta - \frac{1}{\tilde{r}}\partial_{\tilde{r}}\theta = -\dot{\tilde{T_e}} \quad \theta(\tilde{r}=1,t)=0 \tag{1}$$

The homogeneous part is solved by posing $\theta(\tilde{r},\tilde{t}) = \alpha(\tilde{t})\cdot\beta(\tilde{r})$, so

$$\frac{\dot{\alpha}}{\alpha} = \frac{\ddot{\beta}}{\beta} + \frac{\dot{\beta}}{r\beta} = -\lambda^2$$

Solving both ODE yields $\alpha = c_1 e^{-\lambda^2\tilde{t}}$ and $\beta = c_2 J_0(\lambda\tilde{r}) + c_3 Y_0(\lambda\tilde{r})$. The radial part β should remain finite in $r=0$, so $c_3=0$. The other boundary condition leads to

$$J_0(\lambda_n) = 0$$

The non-homogeneous equation is solved by posing

$$\theta = \sum_{n=1}^{\infty} c_n(t) J_0(\lambda_n\tilde{r})$$

Substituting back into the PDE and making use of the homogeneous equation yields

$$\sum_{n=1}^{\infty}[\dot{c}_n + \lambda_n^2 c_n] J_0(\lambda_n\tilde{r}) = -\dot{\tilde{T_e}}$$

Then, we project the equation on $J_0(\lambda_m\tilde{r})$:

$$\sum_{n=1}^{\infty}[\dot{c}_n + \lambda_n^2 c_n]\int_0^1 \tilde{r} J_0(\lambda_n\tilde{r}) J_0(\lambda_m\tilde{r}) d\tilde{r} = -\dot{\tilde{T_e}}\int_0^1 \tilde{r} J_0(\lambda_m\tilde{r}) d\tilde{r}$$

Using the identities seen in class for the integrals of bessel functions, we finally obtain

$$\dot{c}_m + \lambda_m^2 c_m = -\frac{2\dot{\tilde{T_e}}}{\lambda_m J_1(\lambda_m)}$$

When the external forcing temperature is $T_e(t) = T_0[1+\gamma\sin(\omega t)]$, $\dot{\tilde{T_e}} = \gamma\tilde{\omega}\cos(\tilde{\omega}\tilde{t})$. We guess the solution $c_m = a_m\sin(\tilde{\omega}\tilde{t}) + b_m\cos(\tilde{\omega}\tilde{t})$, then we substitute back into the ODE for c_m and match the coefficients of the trig functions, so

$$c_m(\tilde{t}) = \frac{-2\gamma\tilde{\omega}}{\lambda_m J_1(\lambda_m)[\tilde{\omega}^2 + \lambda_m^4]}\left[\tilde{\omega}\sin(\tilde{\omega}\tilde{t}) + \lambda_m^2\cos(\tilde{\omega}\tilde{t})\right]$$

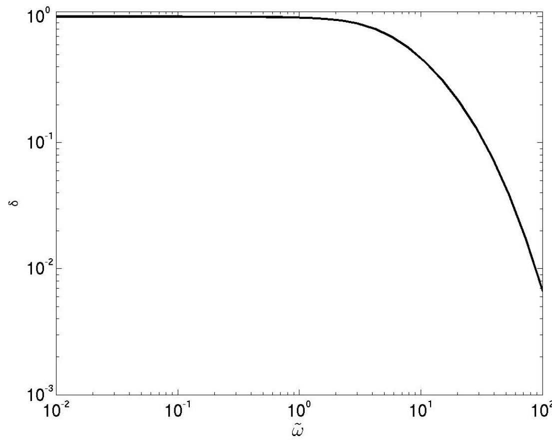

Figure 1: Damping factor δ as a function of $\tilde{\omega}$

On the axis of the cylinder,

$$\tilde{T}(0,\tilde{t}) = \tilde{T}_e(\tilde{t}) + \theta(0,\tilde{t}) = \tilde{T}_e + \sum_{n=0}^{\infty} c_n(\tilde{t})$$

The function $\tilde{T}(0,\tilde{t})$ is computed numerically. It is a sinusoid of amplitude $\delta\gamma$, where δ is defined as a kind of damping factor, measuring how dampened is the external thermal oscillation within the cylinder. Thid damping factor is shown in Fig.1 as a function of $\tilde{\omega}$.

In order to get $\delta < 0.1$ (10%), we should have $\tilde{\omega} > 34.24$. The numerical application gives $\omega = 2\pi/86400$ rad/s, so R should be greater than 19.4cm.

2 The sound of timpani (40 pts)

As suggested, we calculate the natural frequencies of a square elastic membrane. The deflection $w(t,x,y)$ of the membrane obeys the wave equation

$$c^2\nabla^2 w = \partial_{tt} w$$

where c is related to the physical properties of the membrane (tension, density). The boundary conditions are $w = 0$ along all edges.

We seek a solution $w(x,y,t) = X(x)\cdot Y(y)\cdot T(t)$, so

$$\frac{\ddot{X}}{X} + \frac{\ddot{Y}}{Y} = \frac{1}{c^2}\frac{\ddot{T}}{T}$$

Every fraction should be constant, and only trig functions can be involved for the spatial parts X and Y because of the boundary conditions. So

$$\frac{\ddot{X}}{X} = -\alpha^2 \Rightarrow X = A\sin(\alpha_m x), \text{ where } \alpha_m = \frac{m\pi}{L}, m \in \mathbb{N}$$

$$\frac{\ddot{Y}}{Y} = -\beta^2 \Rightarrow Y = B\sin(\beta_n y), \text{ where } \beta_n = \frac{n\pi}{L}, n \in \mathbb{N}$$

then

$$\frac{\ddot{T}}{T} = -c^2(\alpha_m^2 + \beta_n^2) \Rightarrow T = C_1 \cos \omega_{mn} t + C_2 \sin(\omega_{mn} t)$$

where

$$\omega_{mn} = \frac{\pi c}{L}\sqrt{m^2 + n^2} = \omega_0 \sqrt{m^2 + n^2}$$

The frequencies of the various eigenmodes are not integer multiples of a fundamental frequency, so they cannot be called harmonics. The resulting signal $w(x, y, t)$ is not periodic in time, unless only a finite number of modes are selected. The sound created by squared timpani is not nice to hear, it looks like a noise, it cannot be related to a note.

On the other side, as seen in class, round timpani generate a nice sound, where all the natural frequencies are multiple of a fundamental frequency, so a note is heard. The note depends on the size and the tension in the membrane.