18.075 - Pset 4: About solving PDEs to preserve chocolate and make nice music

Due in class (Room 2-135), Friday May 7, 2010 at 2.00pm

You may and should both discuss the problem with other students and read things everywhere you want, but you must write solutions entirely on your own. Also, on the first page of your homework, please do list both the individuals with whom you discussed the homework and the sources that you consulted.

1 The insulating properties of a cylinder (60 pts)

What is worse than to see some delicious belgian chocolate melting then badly crystallizing because the outside temperature exceeds 80^{o}F at noon?!? This should never happen again... (;-) We could buy a fridge, but it's expensive... So let's put this invaluable chocolate at the center of a large insulating (e.g. wooden) cylinder of thermal diffusivity κ and radius R. We suppose that the length of the cylinder is infinite and its radius R is much larger than the size of the chocolate, so we can say that the chocolate is in $r = 0$ where r is the radial coordinate. In this problem, you will be asked to quantify the insulating properties of the cylinder. More exactly, you will have to find the temperature on the central axis when the external temperature experiences daily oscillations.

The temperature field $T(r,t)$ inside the cylinder obeys the heat equation

$$\frac{1}{\kappa}\partial_t T = \partial_{rr} T + \frac{1}{r}\partial_r T \tag{1}$$

The temperature $T(R,t)$ on the surface of the cylinder $r = R$ is equal to the external temperature $T_e(t)$.

We can make the problem (differential equation + boundary condition) dimensionless by using R, R^2/κ and T_0 as the characteristic length, time and temperature respectively, where T_0 is the time-averaged value of $T_e(t)$.

$$\tilde{r} = \frac{r}{R}, \quad \tilde{t} = \frac{t\kappa}{R^2}, \quad \tilde{T} = \frac{T - T_0}{T_0}, \quad \tilde{T}_e = \frac{T_e - T_0}{T_0} \tag{2}$$

so we obtain

$$\partial_{\tilde{t}}\tilde{T} = \partial_{\tilde{r}\tilde{r}}\tilde{T} + \frac{1}{\tilde{r}}\partial_{\tilde{r}}\tilde{T}, \quad \tilde{T}(\tilde{r} = 1, \tilde{t}) = \tilde{T}_e(\tilde{t}) \tag{3}$$

1. This problem consists of an homogeneous equation with a non-homogeneous boundary condition, which is not convenient for finding the eigenfunctions. So we define $\theta(\tilde{r},\tilde{t}) = \tilde{T}(\tilde{r},\tilde{t}) - \tilde{T}_e(\tilde{t})$. Find the differential equation and boundary condition in terms of θ.

2. Solve the homogeneous part of this equation and find the eigenvalues λ_n and eigenfunctions $y_n(\tilde{r})$ that satisfy the boundary conditions.

3. Then, solve the non-homogeneous part, by assuming

$$\theta = \sum_{n=1}^{\infty} c_n(\tilde{t}) y_n(\tilde{r})$$

   Find $c_n(\tilde{t})$ when $T_e(t) = T_0[1 + \gamma \sin(\omega t)]$. Only focus on the long-term solution (neglect the transient). Your answer should involve the dimensionless group $\tilde{\omega} = \omega R^2/\kappa$.

4. Calculate numerically the ratio between the amplitude of the thermal oscillations on the axis ($r = 0$) and the amplitude outside γT_0, as a function of $\tilde{\omega}$. Graph your numerical solution. From there, calculate the critical value of $\tilde{\omega}$ above which the thermal amplitude in the core is less than 10% of the thermal amplitude outside. What thickness R of wood ($\kappa = 8 \cdot 10^{-8}$) would you need to "damp" the daily variations of temperature by 90%?

2 The sound of timpani (40 pts)

Explain why timpani (and drums in general) are round and not squared. To justify your answer, calculate the eigenfrequencies of a square elastic membrane and compare them to the solution for the round elastic membrane seen in class.