18.075 - Pset 3: Solutions

1 A first set of waves

These waves are called Yanai waves, or also mixed Rossby-gravity equatorial waves.

(a) Substituting $(u, v, \eta) = [U(y), V(y), E(y)]e^{i(kx-\omega t)}$ into the set of equations leads to

$$
\begin{aligned}
-i\omega U &= \beta y V - ig'kE & (1)\\
-i\omega V &= -\beta y U - g'\dot{E} & (2)\\
-i\omega E &= -h(ikU + \dot{V}) & (3)
\end{aligned}
$$

To remove U from (1) and (2), we should multiply (1) by βy and (2) by $i\omega$, so

$$(\beta^2 y^2 - \omega^2)V = ig'(\beta k y E + \omega \dot{E}) \tag{4}$$

To remove E from (1) and (2), we should differentiate (1) and multiply (2) by $-ik$, so

$$-(\beta + k\omega)V - \beta y \dot{V} = i(\beta k y U + \omega \dot{U}) \tag{5}$$

Since both (4) and (5) involve the operator $(\omega d/dy + \beta k y)$, we apply this operator to the third equation:

$$-i\omega(\beta k y E + \omega \dot{E}) = -ihk(\beta k y U + \omega \dot{U}) - h(\beta k y \dot{V} + \omega \ddot{V}) \Rightarrow \ddot{V} + \left(\frac{\omega^2 - \beta^2 y^2}{g'h} - \frac{\beta k}{\omega} - k^2\right)V = 0 \tag{6}$$

(b)

$$\frac{d^2 V}{d\tilde{y}^2} + \left(\tilde{\omega}^2 - \tilde{y}^2 - \tilde{k}^2 - \frac{\tilde{k}}{\tilde{\omega}}\right)V = 0$$

(c)

$$\frac{dV}{d\tilde{y}} = \left(\dot{\Phi} - \tilde{y}\Phi\right)e^{-\tilde{y}^2/2}$$

$$\frac{d^2 V}{d\tilde{y}^2} = \left(\ddot{\Phi} - 2\tilde{y}\dot{\Phi} - (1 - \tilde{y}^2)\Phi\right)e^{-\tilde{y}^2/2}$$

so

$$\ddot{\Phi} - 2\tilde{y}\dot{\Phi} + \underbrace{\left(\tilde{\omega}^2 - \tilde{k}^2 - \frac{\tilde{k}}{\tilde{\omega} - 1}\right)}_{2\nu}\Phi = 0$$

This differential equation is called the Hermite equation.

(d) We seek solutions of the form

$$\Phi = \sum_{k=0}^{\infty} A_k \tilde{y}^{k+s}$$

so

$$\tilde{y}\dot{\Phi} = \sum_{k=0}^{\infty} A_k(k+s)\tilde{y}^{k+s}, \quad \ddot{\Phi} = \sum_{k=0}^{\infty} A_k(k+s)(k+s-1)\tilde{y}^{k+s-2} = \sum_{k=-2}^{\infty} A_{k+2}(k+s+2)(k+s+1)\tilde{y}^{k+s}$$

and

$$A_0 s(s-1)\tilde{y}^{s-2} + A_1(s+1)s\tilde{y}^{s-1} + \sum_{k=0}^{\infty}\Big[(k+s+2)(k+s+1)A_{k+2} + 2(\nu-k-s)A_k\Big]\tilde{y}^{k+s} = 0$$

We expect two independent solutions for this second order equation. So in order to remove the terms in $\tilde{y}^{s-2}$ and $\tilde{y}^{s-1}$, we can either set $s=0$ and keep both $A_0 \neq 0$ and $A_1 \neq 0$, or set $s=1$ (resp. $s=-1$) and $A_1 = 0$ (resp. $A_0 = 0$). Let's choose the first option: $s=0$ and $A_0 \neq 0$, $A_1 \neq 0$. Canceling the coefficients of $\tilde{y}^{s+k}$ leads to the recurrence relation

$$A_{k+2} = \frac{2(k-\nu)}{(k+2)(k+1)} A_k, k \in \mathbb{N}^+$$

(e) If $\nu \notin \mathbb{N}$ and $A_0 \neq 0$ (resp. $A_1 \neq 0$), all the coefficients A_{2k} (resp. A_{2k+1}) are different from 0. If we set $2n = k$, the even solution ($A_1 = A_{2k+1} = 0$) can be written as

$$\Phi = \sum_{n=0}^{\infty} A_n \tilde{y}^{2n}, \text{ where } A_{n+1} = \frac{2n-\nu}{(2n+1)(n+1)} A_n$$

Suppose that $|A_n| > |B_n|$ for some large value of n. Then,

$$|A_{n+1}| = \frac{|2n-\nu|}{(2n+1)(n+1)}|A_n| > \frac{|2n-\nu|}{(2n+1)(n+1)}|B_n| = \frac{2|2n-\nu|}{2n+1}|B_{n+1}|$$

Since for large n, $(2n-\nu)/(2n+1) \to 1$, $|A_{n+1}| > 2|B_{n+1}| > |B_{n+1}|$. This means that, as $\tilde{y} \to \infty$, Φ grows faster than $e^{\tilde{y}^2/2}$, so there is no hope that the solution $V = \Phi e^{-\tilde{y}^2/2}$ remains finite far away from the equator. So this solution CANNOT BE PHYSICALLY OBSERVED.

(f) Things change when $\nu \in \mathbb{N}$. Now, the coefficients can be zero as soon as $k = \nu$. Since k is incremented by 2 in the recurrence relation, we should distinguish even and odd values of ν. When ν is even (resp. odd), there are only $(\nu+1)/2$ (resp. $(\nu+2)/2$) non-zero even (resp. odd) coefficients. To avoid a blow-up of the solution in $\tilde{y} \to \infty$, we set $A_1 = 0$ (resp. $A_0 = 0$). The solution Φ is therefore an even (resp. odd) polynomial H_ν of degree ν in $\tilde{y}$, called Hermite polynomial. We are now ensured that V vanishes far from the equator because the decreasing behavior of the exponential dominates the increasing behavior of the polynomial. A consequence of this is that the waves must remain close to the equator, they vanish as soon as $\tilde{y}$ is of the order of unity, which means for distances about $R \sim 250$km from the equator. The five first Hermite polynomials are

$$H_0(\tilde{y}) = 1,\ H_1(\tilde{y}) = \tilde{y},\ H_2(\tilde{y}) = 1 - 2\tilde{y}^2,\ H_3(\tilde{y}) = \tilde{y} - \frac{2}{3}\tilde{y}^3,\ H_4(\tilde{y}) = 1 - 4\tilde{y}^2 + \frac{4}{3}\tilde{y}^4$$

We can also show that the Hermite polynomials are orthogonal to each other, which means that they can serve as a base to represent waves initially composed of many wavenumbers $\tilde{k}$ (cf. Pset 2).

(g) The dispersion relation is therefore

$$\tilde{\omega}^2 - \tilde{k}^2 - \frac{\tilde{k}}{\tilde{\omega}} = 2\nu + 1$$

where $\nu \in \mathbb{N}$. Although it is almost impossible to find an explicit expression $\tilde{\omega}(\tilde{k})$, we can find $\tilde{k}(\tilde{\omega})$ much more easily:

$$\tilde{k} = \frac{-1 \pm \sqrt{1 - 4(2\nu + 1)\tilde{\omega}^2 + 4\tilde{\omega}^4}}{2\tilde{\omega}}$$

For $\nu = 0$, considering that $\tilde{\omega} = -\tilde{k}$ is a spurious solution, the quadratic equation reduces to

$$\tilde{k}_{(\nu=0)} = \tilde{\omega} - \frac{1}{\tilde{\omega}}$$

which is a branch monotonically increasing from $\tilde{\omega} = 0$ in $\tilde{k} \to -\infty$ to $\tilde{\omega} \sim \tilde{k}$ in $\tilde{k} \to +\infty$, and passing through the point $(\tilde{k}, \tilde{\omega}) = (0, 1)$.

For $\nu > 0$, the square root in our expression of $\tilde{k}$ is not defined for

$$\omega \in \left[\sqrt{\nu + 0.5 - \sqrt{\nu(\nu + 1)}}, \sqrt{\nu + 0.5 + \sqrt{\nu(\nu + 1)}}\right]$$

At the boundary points, $d\tilde{\omega}/d\tilde{k} = 0$. The curves are shown in Fig.1. The group velocity goes westwards when $d\tilde{\omega}/d\tilde{k} < 0$ and eastwards when $d\tilde{\omega}/d\tilde{k} > 0$. The lower branch corresponds to Rossby waves while the upper branch corresponds to Poincare waves.

2 The Kelvin waves

(a) The system leads to

$$U = \frac{g'k}{\omega}E = \frac{\omega}{kh}E \text{ and } \dot{E} + \frac{\beta k}{\omega}yE = 0$$

From the first equation, we find the dispersion relation $\omega = \sqrt{g'h}k$, or in dimensionless form $\tilde{\omega} = \tilde{k}$.

(b) The equation for E yields

$$\dot{E} + \frac{y}{R^2}E = 0 \Rightarrow E = E_0 e^{-(y/R)^2}$$

Note that, if the dispersion relation $\omega = -\sqrt{g'h}k$ is considered, then the equation for E is

$$\dot{E} - \frac{y}{R^2}E = 0 \Rightarrow E = E_0 e^{(y/R)^2}$$

which does not vanish in $y \to \pm\infty$. That's the reason for which $\tilde{\omega} = -\tilde{k}$ is a spurious solution of the system.

3 Applications

(a) On these figures, the abscissa is k, the zonal wavenumber and the ordinate is ω, the frequency of the waves. The curves on these figures correspond to the dispersion relation of the equatorial waves, that we have just calculated. On the left figure are represented the waves that are symmetric in y, i.e. the Kelvin waves and the Rossby-gravity waves when ν is even (so the polynomial is even as well). On the right figure are represented the waves that are antisymmetric in y, so the Rossby-gravity waves when ν is odd. Different curves are drawn for a same mode, since it depends on the relatively unknown depth h of the upper layer. According to the experimental data, the Kelvin waves and the Rossby-gravity waves with $\nu = 0$ or 1 are the main waves encountered in the equatorial Pacific.

(b) The group velocity of a Kelvin wave is $d\omega/dk = \sqrt{g'h} \sim 1.4$m/s. Indonesia is about 150°E and Peru about 80°W, so the distance between both is about $(130/360)2\pi R_e \simeq 14500$km. So it takes about 4 months to the Kelvin wave to propagate from Indonesia to Peru. That's why fishing in Peru is affected around Christmas time, so the name "El Niño"...

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Fig. 1