# 18.075 - Pset 3: Waves at the equator: how to explain El Niño

Due in class (Room 2-135), Wednesday Apr. 21, 2010 at 2.00pm

You may and should both discuss the problem with other students and read things everywhere you want, but you must write solutions entirely on your own. Also, on the first page of your homework, please do list both the individuals with whom you discussed the homework and the sources that you consulted.

## 1 Introduction (to better understand what's going on)

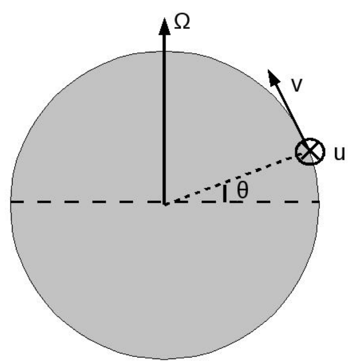



Figure 1: Notations

In this problem set, we focus on some oceanic waves that travel close to the equator. These waves only involve the upper layer of the ocean, from the surface to a depth $h \sim 100$m. Because the ocean is much larger horizontally than deep, we assume that the physical quantities (velocity, pressure, temperature, etc.) do not vary with the depth, but only with the horizontal position. So we propose the following 2-dimensional model

$$\begin{aligned} \partial_t \vec{u} &= \vec{F_c} - g' \vec{\nabla} \eta \\ \partial_t \eta &= -h \vec{\nabla} \cdot \vec{u} \end{aligned}$$

The first equations represent the Newton law, while the second equation ensures the conservation of mass. The vector $\vec{u}$ is the horizontal velocity in this upper layer, averaged vertically over the depth $h$. Locally, the sea surface can be slightly above or below the mean sea-level, so $\eta$ represents the relative elevation of the interface. Note that $\eta$ is not more than a few meters, so $\eta \ll h$. In order to take the lower depths into account, a reduced gravity $g' = g\frac{\rho_2 - \rho_1}{\rho_2}$ is considered instead of gravity (in the equatorial Pacific, $g' \sim 0.02\text{m/s}^2$). The term $\vec{F_c}$ represents the horizontal component of the Coriolis acceleration $-2\vec{\Omega} \wedge \vec{u}$, due to the spin of the Earth (rotation vector $\vec{\Omega}$ with $\Omega = 2\pi/(1day) = 7.2 \cdot 10^{-5}\text{rad/s}$).

We can project both equations on the directions $x$ (the zonal direction, parallel to the equator, pointing Eastwards) and $y$ (the meridional direction, perpendicular to the equator, pointing Northwards), as defined in Fig.1. We can show that the horizontal Coriolis acceleration $\vec{F_c} = [\Omega u \sin\theta, -\Omega v \sin\theta]$ in this system of coordinates, where $\theta$ is the latitude. So the horizontal Coriolis force vanishes at the equator ($\theta = 0$). Since we are only interested in waves close to the equator, we can approximate $\sin(\theta) \simeq y/R_e$, where $y$ is thus the distance to the equator and $R_e \simeq 6370$km the radius of the Earth at the equator. If we define $\beta = 2\Omega/R_e \simeq 2.3 \cdot 10^{-11}\mathrm{m}^{-1}\mathrm{s}^{-1}$ and project the first equation on $x$ and $y$, we obtain

$$\begin{aligned} \partial_t u &= \beta y v - g' \partial_x \eta \\ \partial_t v &= -\beta y u - g' \partial_y \eta \\ \partial_t \eta &= -h(\partial_x u + \partial_y v) \end{aligned} \tag{1}$$

## 2 A first set of waves (60 pts)

First, we look for a solution in terms of the normal modes $(u, v, \eta) = [U(y), V(y), E(y)]e^{i(kx-\omega t)}$. The wavenumber $k$ represents a wave vector in the West-East ($x$) direction.

(a) Rewrite the system (1) in terms of $U$, $V$, $E$ and their derivatives according to $y$. Solve the system for $V$ and show that

$$\frac{d^2V}{dy^2} + \left(\frac{\omega^2 - \beta^2 y^2}{g'h} - \frac{\beta k}{\omega} - k^2\right)V = 0 \tag{2}$$

[Hint: to solve the system, start with the two first equations and turn them into one equation without U and one equation without V. Then, apply the operator $(\omega d/dy + \beta k y)$ to the third equation.]

(b) We define a radius of deformation $R$ such as $R^4 = \frac{g'h}{\beta^2}$. With the numerical values considered here, this radius is of the order of 250km. We set $\tilde{y} = y/R$, $\tilde{k} = kR$ and $\tilde{\omega} = R\omega/(\sqrt{g'h})$. Write Eq.(2) in terms of these tilded dimensionless variables.

(c) Define $V = \Phi e^{-\tilde{y}^2/2}$ and show that $\Phi(\tilde{y})$ obeys to

$$\ddot{\Phi} - 2\tilde{y}\dot{\Phi} + 2\nu\Phi = 0 \tag{3}$$

where $\dot{\Phi} = d\Phi/d\tilde{y}$. What is the value of $\nu$ in terms of $\tilde{\omega}$ and $\tilde{k}$?

(d) Solve Eq.(3) with the method of Frobenius (i.e. expand $\Phi$ in a Laurent series and find the recurrence relation between the coefficients).

(e) Let's suppose first that $\nu \notin \mathbb{N}$. How many coefficients are not zero? Show that there is an even solution (odd coefficients = 0) that can be written as

$$\Phi = \sum_{n=0}^{\infty} A_n \tilde{y}^{2n}, \text{ where } A_{n+1} = \frac{2n - \nu}{(2n+1)(n+1)} A_n$$

Compare the terms of large $n$ of this series with the ones from the Taylor series of

$$e^{\tilde{y}^2/2} = \sum_{n=0}^{\infty} B_n \tilde{y}^{2n} \text{ where } B_{n+1} = \frac{1}{2(n+1)} B_n$$

In particular, prove that if $|A_n| > |B_n|$, then $|A_{n+1}| > |B_{n+1}|$ when $n$ is large. Deduce the behavior of $V$ when $\tilde{y}^2 \to \infty$, i.e. far from the equator.

(f) Let's now suppose that $\nu \in \mathbb{N}$. How many coefficients are not zero? In that case, what is the behavior of $V$ far from the equator. What is the consequence on the location of these waves? Calculate $\Phi$ for all integers $\nu$ from 0 to 4.

(g) Sketch the dimensionless dispersion relation $\tilde{\omega}(\tilde{k})$ for $\nu = 0$, $\nu = 1$ and $\nu = 2$. We suppose that $\tilde{\omega} > 0$, while $\tilde{k}$ can be both positive or negative. Add some arrows to indicate the direction (West or East) of the dimensionless group velocity $\tilde{c}_g = \frac{d\tilde{\omega}}{d\tilde{k}}$ for various values of $\tilde{k}$. Note that, for $\nu = 0$, $\tilde{\omega} = -\tilde{k}$ is a spurious (not physical) solution that should be rejected.

# 3 The Kelvin waves (20 pts)

Next, we look to solutions for which $V(y) = 0$, so $(u, v, \eta) = [U(y), 0, E(y)]e^{i(kx-\omega t)}$. These solutions are called Kelvin waves.

(a) Solve the system (1) according to $E$ and find the dispersion relation $\omega(k)$, then make it dimensionless by using the tilded variables defined above. Sketch the dimensionless dispersion relation on your previous graphic.

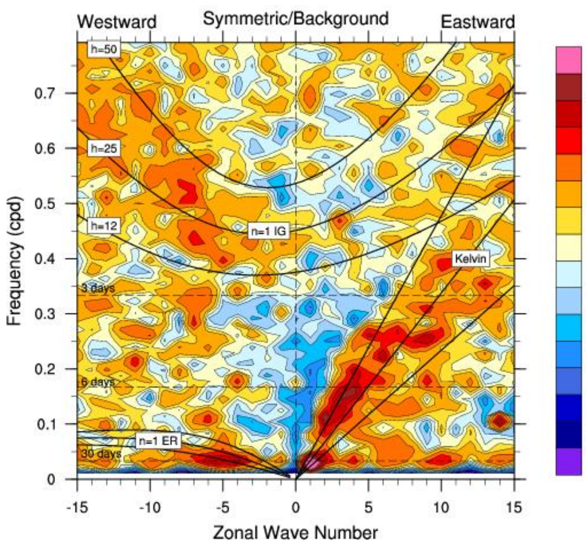
(b) You should also obtain a first order differential equation in $E$ that you have to integrate. Prove by the way that $\tilde{\omega} = -\tilde{k}$ is indeed a non-physical solution.

# 4 Applications (20 pts)

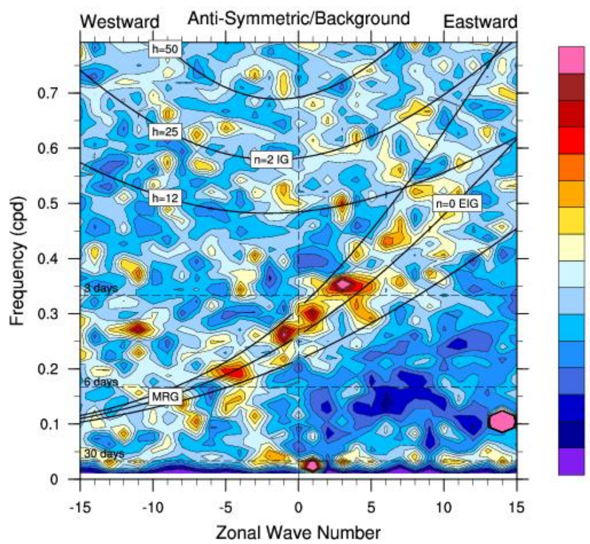
(a) The contours in figure 2 represent some spectral measurements of $\eta$. High peaks correspond to dominant wavelengths/frequencies. Considering all the results you have obtained so far, explain these graphics as much as you can.

Usually each year around the equator, the westwards trade winds progressively move water to the west coasts of the Pacific (Indonesia). Because these waters stay for a long time at the surface of the ocean, they start heating. In the same time, cool waters from the deep ocean comes up to the surface on the east coast (Peru). These waters, very rich in nutrients, attract a lot of fish and make the peruvian fisheries happy. But every two to ten years around July comes El Niño: the trade winds suddenly vanish and the excess of water in Indonesia starts propagating eastwards as a Kelvin wave. As hot water evaporates faster, it starts to rain in the fishless Peru while Indonesia starts experiencing drought and forest fires.

(b) Estimate when in the year the hot waters invade the peruvian coast (use the numerical values found in the text above for $g'$ etc.). From your estimate, explain why the phenomenon is called "El Niño" in spanish.




(a)




(b)

Figure 2: Spectral measurements of the elevation of the ocean in the equatorial Pacific.