18.075 - Pset 2: Solutions

1 A method for integrating exponential of functions

This method is known under the name "Method of steepest descent" or "saddle-point approximation", and is often encountered in physics (e.g. in statistical mechanics).

(a) Suppose that the N maxima are located in $x = x_n$, $n \in [1, N]$. Then the integral becomes

$$I(\lambda) \simeq \sum_{n=1}^{N} e^{\lambda f(x_n)} \sqrt{\frac{2\pi}{-\lambda f''(x_n)}}$$

Nevertheless, if the maxima have different values $f(x_n)$, then only the largest needs to be considered for the approximation. For example, with $\lambda = 20$ and $f(x_2) = 0.8 f(x_1)$, the term in x_2 will be about 50 times smaller than the term in x_1.

(b)

$$\Gamma(\lambda) = \int_0^{+\infty} z^{\lambda-1} e^{-z} dz = \int_0^{+\infty} e^{-z+(\lambda-1)\ln(z)} dz$$

So we can write $\lambda f(z) = (\lambda - 1)\ln z - z$. $f(z)$ is maximum in $z = \lambda - 1$, so $\lambda f(\lambda - 1) = (\lambda - 1)[\ln(\lambda - 1) - 1]$ and $f''(\lambda - 1) = -1/(\lambda - 1)$. By using the approximation formula, we get

$$\Gamma(\lambda) \simeq e^{(\lambda-1)[\ln(\lambda-1)-1]} \sqrt{2\pi(\lambda - 1)} = \sqrt{2\pi}(\lambda - 1)^{(\lambda-1/2)} e^{-(\lambda-1)}$$

If $\lambda = N + 1 \in \mathbb{N}$, then

$$N! = \Gamma(N + 1) \simeq \sqrt{2\pi N}\left(\frac{N}{e}\right)^N$$

so

$$\ln(N!) \simeq N \ln\left(\frac{N}{e}\right) + \frac{1}{2}\ln(2\pi N)$$

for large N. This formula, known as Stirling formula, is very accurate, even for relatively small values of N. Indeed, for $N = 10$, $N! = 3628800$ while its approximation is 3598695, so the relative error is about 0.8%!

(c) The integrand looks like

$$e^{\lambda u(x,y)} e^{i\lambda v(x,y)}$$

so the main contribution to the integral is expected when the argument of the real exponential is maximum. But that's not all... Suppose that this maximum occurs in z_0, in a region where $v(x, y)$ is varying a lot. Then, some of the contributions around z_0 will have $e^{i\lambda v(x,y)} > 0$ while others will have $e^{i\lambda v(x,y)} < 0$. So all these contributions might balance each other if $v(x, y)$ is not relatively constant, and the approach of neglecting every contribution far away from the maximum is not valid anymore.

(d) According to the Cauchy-Riemann equations, $\partial_x v = -\partial_y u$ and $\partial_y v = \partial_x u$ so every point where $\vec{\nabla} u = 0$ is also a point where $\vec{\nabla} v = 0$. For the second derivative, $\partial_x(\partial_x u) = \partial_x(\partial_y v)$ and $\partial_y(\partial_y u) = -\partial_y(\partial_x v)$. So the trace of the Hessian matrix is $\partial_{xx} u + \partial_{yy} u = 0$, and the corresponding eigenvalues can only have opposite signs. Every point where $\vec{\nabla} u = 0$ is therefore a saddle. The same holds for v.

(e) The Taylor series around z_0 writes

$$f(z) = f(z_0) + \frac{1}{2} s^2 \rho e^{i(2\theta+\varphi)}$$

so $u(x, y) = u(x_0, y_0) + \frac{1}{2} s^2 \rho \cos(2\theta + \varphi)$ and $v(x, y) = v(x_0, y_0) + \frac{1}{2} s^2 \rho \sin(2\theta + \varphi)$. Since we want v to be constant and u to be maximum around (x_0, y_0), we need $2\theta + \varphi = \pi$, so

$$\theta = \frac{\pi - \varphi}{2}$$

and locally, on the best path,

$$f(z) = f(z_0) - \frac{1}{2} s^2 \rho$$

(f) The integral becomes

$$I(\lambda) \simeq e^{\lambda f(z_0)} \int_C e^{-\frac{\lambda \rho s^2}{2}} dz$$

We can extend our local path in s to infinity on both sides, and say that $dz = e^{i\theta} ds$ so

$$I(\lambda) \simeq e^{\lambda f(z_0)} \int_{-\infty}^{+\infty} e^{-\frac{\lambda \rho s^2}{2}} e^{i\frac{\pi}{2}} e^{-i\frac{\varphi}{2}} ds = i e^{\lambda f(z_0)} \sqrt{\frac{2\pi}{\lambda |f''(z_0)|}} e^{-i\frac{\varphi}{2}} = e^{\lambda f(z_0)} \sqrt{\frac{2\pi}{-\lambda f''(z_0)}}$$

2 The linear and forced Korteweg-de Vries equation

(a) By substituting Laplace transforms of both f and h in the differential equation, we obtain

$$\int_{-\infty}^{+\infty} \Big[(-i\omega_0 + \alpha i k - \beta i k^3) H(k) - F(k) \Big] e^{i(kx - \omega_0 t)} dk = 0$$

so

$$H(k) = \frac{F(k)}{i(\alpha k - \beta k^3 - \omega_0)} = \frac{F(k)}{i[\Omega(k) - \omega_0]}$$

(b) The integrand blows up for $\Omega(k_n) = \omega_0$. The graph of $\Omega(k)$ is shown in Fig.1. It's a cubic function with three zeros in $-\sqrt{\alpha/\beta}$, 0 and $\sqrt{\alpha/\beta}$, separated by one minimum and one maximum. So for every $\omega_0 > 0$, we expect at least one negative value k_1. Two positive values k_2 and k_3 are also obtained as long as $\omega_0^2 < 4\alpha^3/(27\beta)$.

(c) Now, the singularities $\tilde{k}_n$ obey to

$$\omega_0 + i\varepsilon = \Omega(\tilde{k}_n) = \Omega(k_n + i\delta) \simeq \Omega(k_n) + i\delta \Omega'(k_n)$$

so

$$\delta = \frac{\varepsilon}{c_g(k_n)}$$

The sign of δ is thus given by the sign of $c_g(k_n) = \Omega'(k_n)$. Therefore, both the first and the third roots move below the real axis while only the second root $\tilde{k}_2$ is now located above the real axis.

(d) In order to use the Cauchy's residue theorem, we should find a contour on which the exponential e^{ikx} goes to 0 as $x \to \infty$ or $x \to -\infty$. On the semi-circle above (resp. below) the real axis, the imaginary part of k is positive (resp. negative), so the corresponding exponential vanishes (resp. blows up) as $x \to +\infty$ and blows up (resp. vanishes) as $x \to -\infty$. Therefore, the semi-circle above works for $x \to +\infty$ and the semi-circle below for $x \to -\infty$.

(e) As $\tilde{k}_2$ is the only singularity inside the upper semi-circle, we need to calculate the residual in $\tilde{k}_2$, which is a single pole:

$$\mathrm{Res}(\tilde{k}_2) = \lim_{k \to \tilde{k}_2} \frac{F(k) e^{i(kx - \omega_0 t)}(k - \tilde{k}_2)}{i[\Omega(k) - \omega_0 - i\varepsilon]} = \frac{F(\tilde{k}_2) e^{i(\tilde{k}_2 x - \omega_0 t)}}{i\Omega'(\tilde{k}_2)}$$

Since the integral on the circular part of the contour vanishes as $x \to \infty$,

$$\lim_{x \to \infty} \tilde{h}(x,t) = -2\pi i \mathrm{Res}(\tilde{k}_2) = -2\pi \frac{F(\tilde{k}_2) e^{i(\tilde{k}_2 x - \omega_0 t)}}{\Omega'(\tilde{k}_2)}$$

The solution $h(x,t)$ of Eq.(1) is obtained by taking the limit for $\varepsilon \to 0$, so

$$\lim_{x \to \infty} h(x,t) = -2\pi i \mathrm{Res}(k_2) = -2\pi \frac{F(k_2) e^{i(k_2 x - \omega_0 t)}}{\Omega'(k_2)}$$

3 Waves in the wake of a ship (25 pts)

(a) The integral that gives $h(\vec{x}, t)$ involves the exponential of a function of s, and the argument of this exponential goes to infinity as $\vec{x}$ does. So we can use the method of steepest descent established in problem 1 to approximate the integral. To do so, we should ensure that the imaginary part of the exponential is constant, so we want $\Psi(s) = \vec{k}(s) \cdot \vec{d}$ to be constant.

$$\frac{d\Psi}{ds} = \frac{d\vec{k}}{ds} \cdot \vec{d} = 0$$

But since $d\vec{k}/ds$ is already perpendicular to $\vec{c}_g(\vec{k})$, we deduce that $\vec{c}_g(\vec{k})$ is parallel to $\vec{d}$. In other words, only the wavenumbers $\vec{k}(s)$ for which the group velocity $\vec{c}_g[\vec{k}(s)]$ is parallel to $\vec{d}$ will be observed in $\vec{x}$. This result generalizes what was shown in Problem 2 (only the singularities for which c_g has the same sign as x have to be taken into account).

(b) The group velocity is given by

$$\vec{c}_g = \vec{\nabla}_{\vec{k}} \Omega = \left(\frac{\sqrt{g} k_1}{2k^{3/2}} - V, \frac{\sqrt{g} k_2}{2k^{3/2}} \right) = \frac{V}{2} \left(\cos^2\theta - 2, \sin\theta \cos\theta \right)$$

The direction of $\vec{c}_g$ makes an angle φ with the direction of the ship, where

$$\tan\varphi = \frac{\sin\theta\cos\theta}{\cos^2\theta - 2}$$

By differentiating $\tan\varphi$ according to θ, we can show that the maximum value of $\tan\varphi$ occurs for $\cos\theta = \sqrt{2/3}$, from which we deduce $\sin\varphi = 1/3$, so $\varphi = 19.5^{\mathrm{o}}$. Since the waves are only observed in the directions taken by $\vec{c}_g$, no waves are observed outside the cone of half-angle 19.5^{o}.

Fig. 1

$$\Omega(k) = \alpha k - \beta k^3$$