# 18.075 - Pset 1: Cauchy-Riemann equations in fluid dynamics

Due in class (Room 2-135), Monday Feb. 22, 2010 at 2.00pm

You may and should both discuss the problem with other students and read things everywhere you want, but you must write solutions entirely on your own. Also, on the first page of your homework, please do list both the individuals with whom you discussed the homework and the sources that you consulted.

## 0 Introduction (to better understand what's going on...)

In fluid dynamics, some large-scale steady and incompressible flows (e.g. around a plane) may be efficiently analysed by using functions of a complex variable. These flows, for which the effects of viscosity can be neglected at first, are said ideal.

For a 2D velocity field $\vec{u}(x,y)$ of components $u(x,y)$ and $v(x,y)$, the condition of incompressibility writes

$$\vec{\nabla} \cdot \vec{u} = \partial_x u + \partial_y v = 0 \tag{1}$$

Because the divergence of a curl is always 0, we can say that $\vec{u}$ should be the curl of something. Indeed, we can find a function $\Psi(x,y)$, the stream function, that obeys

$$u = \partial_y \Psi \text{ and } v = -\partial_x \Psi \tag{2}$$

and so satisfies Eq.(1). The iso-$\Psi$ curves are always perpendicular to the gradient $(\partial_x \Psi, \partial_y \Psi)$ of $\Psi$, so they are also parallel to the velocity field $\vec{u}$, they coincide with the streamlines of the flow.

If in addition, there is no vorticity, the flow is said potential (note that this is more restrictive than ideal). The vorticity $\vec{\omega} = \vec{\nabla} \wedge \vec{u}$ is the curl of the velocity, and we know that the curl of a gradient is always identically zero. So we can also find another function, called the velocity potential $\varphi$, that obeys

$$\vec{u} = \vec{\nabla}\varphi \Rightarrow u = \partial_x \varphi \text{ and } v = \partial_y \varphi \tag{3}$$

The iso-$\Phi$ are perpendicular to the velocity field, so to the streamlines.

Since both functions $\Psi$ and $\varphi$ describe the same velocity, we write

$$\partial_x \varphi = \partial_y \Psi \text{ and } \partial_y \varphi = -\partial_x \Psi \tag{4}$$

These equations are exactly similar to the Cauchy-Riemann equations seen in lecture, which suggest to define a complex function

$$\Phi(z) = \varphi(x,y) + i\Psi(x,y) \tag{5}$$

called the complex potential. The flow is thus incompressible and irrotational everywhere $\Phi(z)$ is analytic.

## 1 A 2D flow described by a stream function (15 pts)

Given the stream function

$$\Psi = x^2 + ay^2, \quad a \in \mathbb{R}, \tag{6}$$

(a) Sketch every different kinds of streamlines obeying to Eq.(6) in the $(x, y)$-plane for three values of $a$, namely -1, 0 and 1. Make a separate graphic for each value and indicate the direction of the flow on each streamline. Calculate the velocity field $(u, v)$.

(b) By using the Cauchy-Riemann equation, find the only value of $a$ for which the stream function corresponds to a potential flow. For that specific value, determine the complex potential $\Phi(z)$.

(c) Find a relation between $a$ and the vorticity $\omega = \partial_x v - \partial_y u$ of the flow.

## 2 Some flows around angular surfaces (15 pts)

We have seen in lecture that the complex potential

$$\Phi = (Cz)^{m+1}, \quad m \in \mathbb{Z}, C = C_0 e^{-i\alpha} \in \mathbb{C} \tag{7}$$

is an analytic function everywhere. But what flow does it represent?

(a) Find $\varphi$ and $\Psi$ in polar coordinates $(r, \theta)$.

(b) The function $\Psi$ that you should obtain vanishes for several values of $\theta$. On six different graphics, sketch TWO SUCCESSIVE iso-$\Psi = 0$ for $m = -1/2$, -1/4, 0, 1/2, 1 and 2. Take $\alpha = \pi/3$. Then, sketch the only iso-$\varphi = 0$ in between. In the next, we will consider that these two iso-$\Psi$ curves are solid surfaces that set the domain of the flow. Why can we do that ?

(c) Calculate the velocity field $(v_r, v_\theta)$ in polar coordinates.

(d) On the six previous graphics, roughly sketch the streamlines in between your two iso-$\Psi = 0$. Indicate the direction of the flow. The velocity field might be helpful.

## 3 Sources, sinks and vortices (30 pts)

Suppose a function $\Phi(z) = \varphi + i\Psi$, analytic everywhere except in some singular points.

(a) Calculate $w(z) = \mathrm{d}\Phi/\mathrm{d}z$ as a function of $(u, v)$ in cartesian coordinates, then as a function of $(v_r, v_\theta)$ in polar coordinates.

(b) Show that

$$\oint_C w dz = \Gamma + iQ, \text{ where } \Gamma = \oint_C \vec{v} \cdot \vec{t} d\ell \text{ and } Q = \oint_C \vec{v} \cdot \vec{n} d\ell \tag{8}$$

Give a physical interpretation for $\Gamma$ and $Q$.

(c) If $\Phi(z)$ is analytic everywhere, calculate $\Gamma$ and $Q$. Is your result in agreement with some previous hypotheses made on the flow?

For the next of this problem, we consider the complex potential

$$\Phi = (a + ib) \log z, \qquad a, b \in \mathbb{R} \tag{9}$$

(d) Is this potential analytic everywhere? If no, where is it not?

(e) Calculate $w(z)$, then $v_r$ and $v_\theta$.

(f) Calculate $\oint_C wdz$ for a closed contour equal to the unit circle.

(g) Calculate $\oint_C wdz$ for a closed contour described by the equation $z = 5 + e^{i\theta}$, $\theta \in [0, 2\pi]$.

(h) Calculate $\Gamma$ and $Q$ for an arbitrary contour, as functions of the parameters $a$ and $b$.

(i) Sketch some streamlines first in the case $a = 0, b > 0$, then in the case $b = 0, a < 0$, then in the case $b = 0, a > 0$, and last in the mixed case $a < 0, b > 0$.

## 4 Flow around the airfoil of a plane (40 pts)

We have seen in lecture that a conformal mapping defined by $Z = f(z)$ preserves angles everywhere $f(z)$ is analytic. Here we consider the mapping

$$Z = z + \frac{a^2}{z} \tag{10}$$

and first the complex potential

$$\Phi(Z) = UZ = U\left(z + \frac{a^2}{z}\right). \tag{11}$$

(a) Represent the streamlines in the $Z$-plane. What is the physical meaning of $U$?

(b) Suppose a circle of radius $R$ in the $z$-plane. How is this circle transformed in the $Z$-plane, depending on $a$? Show that it becomes a line (a plane in 3D) when $a = R$. What are the boundaries of this line?

(c) Visibly, the flow obtained in (a) could perfectly match around the line corresponding to $a = R$. So the streamlines corresponding to $\Phi(z)$ in the $z$-plane should correspond to the flow around a cylinder of radius $R$. Sketch these streamlines. Calculate $\Psi$ and $\varphi$, then $v_r$ and $v_\theta$. Show that there are two stagnation points on the surface of the circle of radius $R$ where the velocity vanishes.

Now, we complexify the potential a little bit:

$$\Phi(z) = U\left(ze^{-i\alpha} + \frac{R^2}{ze^{-i\alpha}}\right) + i\frac{\Gamma}{2\pi}\log\left(\frac{ze^{-i\alpha}}{R}\right) \tag{12}$$

(d) What is the physical meaning of these two additions, namely the factor $e^{-i\alpha}$ and the logarithm term?

(e) Calculate $v_r$ and $v_\theta$ (e.g. from $w(z)$ as suggested in problem 4a). Again find the two stagnation points. Then, roughly sketch the streamlines in the $z$-plane (without actually calculating them). Do these points always exist?

Now, let's come back to the $Z$-plane. We have seen that the circle in $z$ is transformed into a line in $Z$. But thanks to our two additions in $\Phi(z)$, the flow described by $\Phi(Z)$ is a little bit more complex than the one obtained in (a).

(f) By using the chain rule, calculate $W = \mathrm{d}\Phi/\mathrm{d}Z$. Keep $z$ instead of $Z$ in your answer.

(g) Calculate $W(\theta)$ on the line and deduce the cartesian velocity field $(u, v)$. Roughly sketch the streamlines and indicate the stagnation points on the line.

(h) The german physicist M.W. Kutta showed that the flow is stable when one of the stagnation points is located at the right-end of the line. What is the corresponding value of $\theta$? Find the only value of the circulation $\Gamma$ that satisfies Kutta's condition.

(i) The lift force exerted by an air flow on an object is given by $F_L \simeq \rho_a U \Gamma$ (per unit of span length), where $U$ is the velocity of the flow and $\rho_a$ the density of air. By assuming that the wings of a plane are thin sheets in first approximation, calculate the lift as a function of the angle of attack of the flow.

(j) Explain why fighters usually take-off with a higher angle of attack than a Boeing 747.

## 5 Bonus (10 additional points)

Any idea about other problems in physics that would directly benefit from analytic functions and conformal mapping? Note that this open question does not require more text and clever things than the equivalent of 10 points elsewhere in the problem set...