18.075 - Practice exam 2 - Friday May 7, 2010 - 2.00pm

Your name:

No calculators, books, notes or cell phones may be used. Show all your work on these sheets: if you need extra space, you can write on the backs of pages.

1. (20 pts) Quick answers (no justification needed)

   - Legendre series representation of $3x^2$

     $2P_2(x) + P_0(x)$

   - $\int_{-\infty}^{+\infty} \sin(3x)\cos(4x)dx$

     0

   - Orthogonality relation for eigenfunctions $y_n(x)$ of $y'' + \lambda e^{2x} y = 0$ on $0 \leq x \leq b$

     $\int_0^b y_n(x) \cdot y_m(x) \cdot e^{2x} dx = 0.$ m

   - Classify as linear, nonlinear or quasi-linear and state order of

     $$x(\partial_x \Phi)^2 + \partial_{yyy}\Phi = x\Phi$$

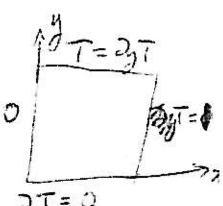
     Quasi - linear, 3^{rd} order

   - Classify as parabolic, hyperbolic or elliptic: $\partial_{xx}\Phi + \partial_{yy}\Phi + 2\partial_{xy}\Phi = x^2y^2$ Parabolic
   - General solution of $x^2y'' + xy' + (x^2 - 4)y = 0$

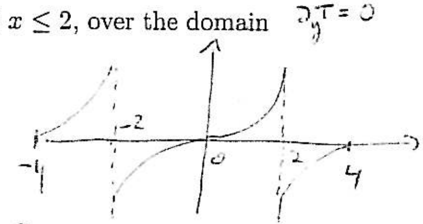
     $A\, J_2(x) + B\, Y_2(x)$

   - If $T(x, y) = X(x).Y(y)$ satisfies $\nabla^2 T = 0$, $x \in [0, 1], y \in [0, 1]$ with the boundary conditions $T(0, y) = 0$, $\partial_y T(x, 0) = 0$, $T(x, 1) = \partial_y T(x, 1)$ and $T(1, y) = 1$, do you first solve for $X(x)$ or for $Y(y)$?

     $Y(y)$ because it's homogeneous



   - Sketch the Fourier sine series representation of x^2, valid over $0 \leq x \leq 2$, over the domain $-4 \leq x \leq 4$



   - What is the weighting function in the following S-L problem ?

     $$y'' + x^3 y + \lambda \frac{1}{\cos(x)} y = 0, \qquad y(a) = y(b) = 0$$

     $w(x) = \frac{1}{\cos x}$

2. (30pts) Solve the Laplace equation $\nabla^2 T(r,\theta) = 0$ outside the circle $r = R$ with the boundary ~~conditions~~ condition $T(R,\theta) = \cos^2(\theta)$.

$$\frac{1}{r}\partial_r\left(r\,\partial_r T\right) + \frac{1}{r^2}\partial_{\theta\theta} T = 0$$

$$T = \cancel{X}(r)\cdot\varphi(\theta) \implies \underbrace{\frac{r^2 X'' + rX'}{X}}_{\omega^2} + \underbrace{\frac{\varphi''}{\varphi}}_{-\omega^2} = 0$$

Angular part $\implies \varphi = A\cos(\omega\theta) + B\sin(\omega\theta)$

Periodic $2\pi \implies \omega = n;\ n \in \mathbb{N} \implies \varphi_n = A_n\cos(n\theta) + B_n\sin(n\theta)$

B.C. even $\implies B_n = 0;\ \forall n \implies \varphi_n = A_n\cos(n\theta)$

Radial part: $r^2 \cancel{X}'' + rX' - n^2 X = 0 \to$ equidimensional equation

$X = r^\alpha \implies \alpha(\alpha-1) + \alpha - n^2 = 0 \implies \alpha = \pm n$ if $n \neq 0$

$\alpha = n$ rejected because r^n not finite when $r \to +\infty$

$\implies$ only keep $X = r^{-n}$

Note: if $n = 0$: $r^2 X'' + rX = 0 \implies X' = \frac{C}{r}$

$\implies X = C\log r + D$

But $X(+\infty)$ finite $\implies C = 0 \implies X =$ cst.

$\to$ General solution: $T = \sum_{n=0}^{\infty} A_n r^{-n}\cos(n\theta)$

Non-homog B.C.: $T(R,\theta) = \sum_{n=0}^{\infty} A_n R^{-n}\cos(n\theta) = \cos^2\theta = \frac{1+\cos 2\theta}{2}$

$\cos(n\theta)$ orthog. to each other $\implies A_0 = \frac{1}{2};\ A_2 = \frac{R^2}{2};\ A_k \neq 0;\ k \neq \{0;2\}$

$$\implies T = \frac{1}{2} + \frac{R^2}{2r^2}\cos(2\theta)$$

3. (20pts) Use the Frobenius method to solve $xy'' + 2y' + xy = 0$. $\longrightarrow y'' + \frac{2}{x} y' + y = 0$

$$y = \sum_{k=0}^{\infty} c_k x^{k+s} \implies \frac{y'}{x} = \sum_{k=0}^{\infty} c_k (k+s) x^{k+s-2}$$

$$= c_0 s x^{s-2} + c_1 (s+1) x^{s-1} + \sum_{k=0}^{\infty} c_{k+2} (k+s+2) x^{k+s}$$

$$y'' = \sum_{k=0}^{\infty} c_k (k+s)(k+s-1) x^{k+s-2} = c_0 s(s-1) x^{s-2} + c_1 (s+1) s x^{s-1} + \sum_{k=0}^{\infty} c_{k+2} (k+s+2)(k+s+1) x^{k+s}$$

$$y'' + \frac{2}{x} y' + y = 0 \implies c_0 s(s+1) x^{s-2} + c_1 (s+1)(s+2) x^{s-1}$$

$$+ \sum_{k=0}^{\infty} \left\{ c_{k+2} (k+s+2)(k+s+3) + c_k \right\} x^{k+s} = 0$$

Take $\boxed{s = -1}$ $\implies c_{k+2} (k+1)(k+2) + c_k = 0$; $k \geq 0$ with c_0, c_1 arbitrary

$$\implies c_{k+2} = \frac{c_k}{(k+1)(k+2)}$$

$$\implies c_k = \begin{cases} \frac{c_0}{k!} & \text{if } k \text{ even} \\ \frac{c_1}{k!} & \text{if } k \text{ odd} \end{cases}$$

$$\implies y = \frac{1}{x} \cdot \left[c_0 \sum_{\substack{k=0 \\ k \text{ even}}}^{\infty} \frac{x^k}{k!} + c_1 \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{x^k}{k!} \right] = \frac{c_0 \cos x + c_1 \sin x}{x}$$

4. (30pts) Find the solution $z(x, y)$ satisfying the PDE

$$\partial_{xx} z - 2\partial_{xy} z - 3\partial_{yy} z = 0$$

and the boundary conditions $z(x, y = 0) = x + 2$ and $z(x, y = x) = 2 - 16x^2$.

$$\left(\frac{dy}{dx}\right)^2 + 2\left(\frac{dy}{dx}\right) - 3 = 0 \Rightarrow \frac{dy}{dx} = -3 \text{ or } 1$$

$$\Rightarrow \begin{cases} u_1 = y + 3x \\ u_2 = y - x \end{cases}$$

$$\Rightarrow z = F(y+3x) + G(y-x)$$

$$z(x, y=0) = F(3x) + G(-x) = x+2$$

$$z(x, y=x) = 2 - 16x^2 = F(4x) + G(0)$$

$$\Rightarrow F(4x) = 2 - G(0) - 16x^2$$

$$\Rightarrow F(x) = 2 - G(0) - x^2$$

$$\Rightarrow F(3x) = 2 - G(0) - 9x^2$$

$$\Rightarrow 2 - G(0) - 9x^2 + G(-x) = x + 2$$

$$\Rightarrow G(-x) = G(0) + x + 9x^2$$

$$\Rightarrow G(x) = G(0) - x + 9x^2$$

$$\Rightarrow z = 2 - G(0) - (y+3x)^2 - (y-x) + 9(y-x)^2$$

$$= 2 - (y+3x)^2 + 9(y-x)^2 - (y-x)$$

$$= 8y^2 - 24xy + x - y + 2$$