18.075 - In-class exam 1 (Make-up) - March 29, 2010 - 2.00pm

Your name:

No calculators, books, notes or cell phones may be used. Show all your work on these sheets: if you need extra space, you can write on the backs of pages.

1. (20 pts) Quick answers. Should take minimal computation. Just write answer. No justification required. 2 points each.

   - $$\oint_\gamma \frac{\cosh(z)}{(z-2)^8}dz$$ $\boxed{0}$ analytic inside γ

     where γ is the unit circle.
   - Classify singularity at ∞ of $z\cosh z$ $\quad z = \frac{1}{t} \Rightarrow \frac{\cosh(\frac{1}{t})}{t} \rightarrow \boxed{\text{essential}}$
   - $$\int_{-\infty}^{+\infty} \frac{\cos^3 z}{z^3+z^9}dz$$ $\boxed{0}$ (odd function)
   - Order of pole of $\frac{\sin^2 z}{z^8}$ at $z=0$ $\boxed{6}$
   - $$\mathrm{Res}\left(z^2e^{3/z^3},0\right)$$ $z^2\left(1+\frac{3}{z^3}+\dots\right) = z^2+\frac{3}{z}+\dots \rightarrow \boxed{3}$
   - Identify and classify singularities of $\log(z^4+z^2)$ $\boxed{\text{Branch points}}$ in $z^4+z^2=0 \Rightarrow \boxed{z=0}$ or $\boxed{z=\pm i}$
   - $$\oint_\gamma \left(\sin z + \cosh z + e^{z^2} + 3z + \frac{4}{z}\right)dz$$

     where γ is the unit circle. $\hookrightarrow 4 \cdot 2\pi i = \boxed{8\pi i}$
   - $$\int_\Gamma 3z^2dz$$ $\longrightarrow$ I forgot to specify the contour

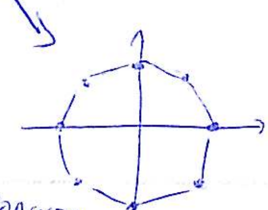
     $\Rightarrow z^3\big]_f - z^3\big]_i$ (= 0 if closed curve)
   - Draw roots of $z^8 = 1$ in complex plane
   - Draw domain of convergence of

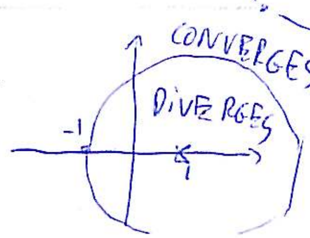
     $$\sum_{n=0}^{\infty}\left(\frac{2}{z-1}\right)^n$$

     in complex plane.

$$\frac{2^{n+1}}{|z-1|^{n+1}} \cdot \frac{|z-1|^n}{2^n} < 1$$

$$\Rightarrow |z-1| > 2$$





2. (5 pts) Where is $f(z) = z^2\Im(z)$ analytic?

$$f(z) = (x+iy)^2 \cdot y = \underbrace{x^2y - y^3}_{u} + \underbrace{2ixy^2}_{iv}$$

Cauchy-Riemann

$$\begin{cases} \dfrac{\partial u}{\partial x} = 2xy \neq \dfrac{\partial v}{\partial y} = 4xy \quad \text{except in } x=0 \text{ or } y=0 \\ \dfrac{\partial u}{\partial y} = x^2 - 3y^2 \neq -\dfrac{\partial v}{\partial x} = -2y^2 \quad \text{except in } x = \pm y \end{cases}$$

$\Longrightarrow$ Nowhere analytic except in $z=0$

3. (10 pts) Given $u(x, y) = 2xy$, find its harmonic conjugate $v(x, y)$ such as $f(z) = (u + iv)$ is an analytic function everywhere. Find $f(z)$ and df/dz in terms of z.

$$u = 2xy \Rightarrow \begin{cases} \dfrac{\partial u}{\partial x} = 2y = \dfrac{\partial v}{\partial y} \Rightarrow v = y^2 + g(x) \\ \dfrac{\partial u}{\partial y} = 2x = -\dfrac{\partial v}{\partial x} \Rightarrow v = h(y) - x^2 \end{cases}$$

$$\Longrightarrow \boxed{v = y^2 - x^2}$$

$$\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 2(y - ix) = -2iz$$

$$\Longrightarrow \boxed{f = -iz^2}$$

4. (15 pts) Find the power series expansion for $f(z)$ about z_0 valid in the prescribed domain. Identify the series as Taylor or Laurent.

- $f(z) = e^{3-z}$, $z_0 = i$ valid for $|z-i| < \infty$

$$t = z - i \Rightarrow f(t) = e^{3-i} e^{-t} = e^{3-i}\left(1 - t + \frac{t^2}{2} - \cdots\right)$$

$$\Rightarrow \boxed{f(z) = e^{3-i} \sum_{k=0}^{\infty} (-1)^k \frac{(z-i)^k}{k!}} \quad (\text{Taylor})$$

- $g(z) = \frac{1}{z(z-1)}$, $z_0 = 1$, valid for $0 < |z-1| < 1$

$$g(z) = \frac{1}{z-1} \cdot \frac{1}{1+(z-1)} = \frac{1}{z-1} \cdot \sum_{k=0}^{\infty} (-1)^k (z-1)^k$$

$$= \boxed{\sum_{k=0}^{\infty} (-1)^k (z-1)^{k-1}} \quad (\text{Laurent})$$

- $h(z) = \frac{1}{z(z-1)}$, $z_0 = 1$, valid for $|z-1| > 1$.

$$h(z) = \frac{1}{z-1} \cdot \frac{1}{1+(z-1)}$$

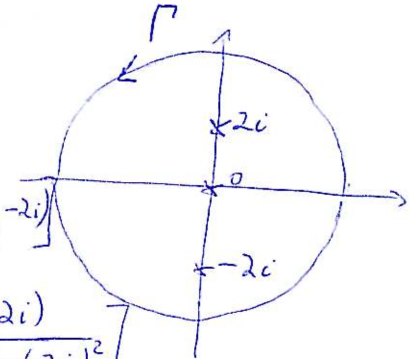
$$= \frac{1}{(z-1)^2} \cdot \frac{1}{1 - \frac{-1}{z-1}}$$

$$= \frac{1}{(z-1)^2} \cdot \sum_{k=0}^{\infty} \frac{(-1)^k}{(z-1)^k} = \boxed{\sum_{k=0}^{\infty} \frac{(-1)^k}{(z-1)^{k+2}}}$$

5. (15 pts) Evaluate

$$\oint_\Gamma \frac{\sin z}{z^2(z^2+4)}dz$$

where Γ is the counterclockwise path about a circle of radius 5 centered on the origin.



Simple poles in $z = 0$; $z = +2i$ and $z = -2i$

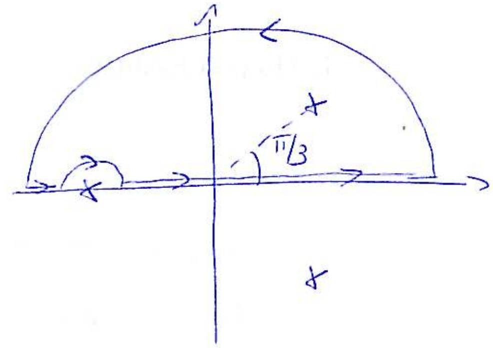
$$\Rightarrow \oint_\Gamma \cdots = 2\pi i \left[Res(\cdots, 0) + Res(\cdots, 2i) + Res(\cdots, -2i) \right]$$

$$= 2\pi i \cdot \left[\frac{1}{4} + \frac{\sin 2i}{4i \cdot (2i)^2} + \frac{\sin(-2i)}{(-4i)\cdot(-2i)^2} \right]$$

$$= \boxed{\frac{\pi i}{2} - \frac{\pi i}{4} \sinh(2)}$$

6. (20 pts) Evaluate

$$\int_{-\infty}^{+\infty} \frac{dx}{x^3+8}$$



Simple poles in $z = 2e^{\frac{i\pi}{3} + \frac{2ki\pi}{3}}$; $k \in \{0;1;2\}$

$= 2e^{i\pi/3}$; -2 ; $2e^{5i\pi/3}$

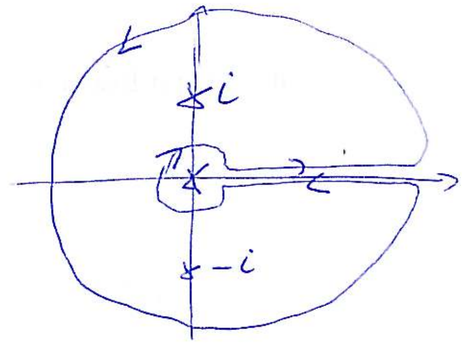
$$\Rightarrow \int_{-\infty}^{+\infty} \frac{dx}{x^3+8} = 2\pi i \, \mathrm{Res}\left(\frac{1}{z^3+8} ; 2e^{i\pi/3}\right) + \pi i \, \mathrm{Res}\left(\frac{1}{z^3+8}, -2\right)$$

$$= \frac{2i\pi}{3 \cdot 4e^{2i\pi/3}} + \frac{i\pi}{3 \cdot (-2)^2} = \frac{i\pi}{6}\left[\cos\left(\frac{-2\pi}{3}\right) + i\sin\left(\frac{-2\pi}{3}\right)\right] + \frac{i\pi}{12}$$

$$= \boxed{\frac{-\pi\sqrt{3}}{12}}$$

7. (15 pts) Evaluate

$$\int_0^\infty \frac{x^{1/2}}{x^2+1}dx$$



Simple poles in $z = \pm i$

Branch point in $z = 0$

$$\int_0^{+\infty} \frac{x^{1/2}\,dx}{x^2+1} = \frac{2\pi i}{1-e^{2\pi i/2}} \cdot \left[\mathrm{Res}\left(\frac{z^{1/2}}{z^2+1}\,;\, \underset{= e^{i\pi/2}}{i}\right) + \mathrm{Res}\left(\frac{z^{1/2}}{z^2+1}\,,\, \underset{e^{3i\pi/2}}{-i}\right)\right]$$

$$= i\pi \cdot \left[\frac{e^{i\pi/4}}{2i} + \frac{e^{3i\pi/4}}{-2i} \right]$$

$$= \frac{\pi}{2}\left[e^{i\pi/4} - e^{3i\pi/4} \right] = \boxed{\frac{\pi\sqrt{2}}{2}}$$