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4.1 Geometrical solution for polynomial roots

1 Sets in the complex plane #02

1.1 Statement: Sets in the complex plane #02

Describe the sets in the complex plane defined by the following conditions:
Now multiply both the numerator and the denominator by $e$.

For any

1.2 Answer: Sets in the complex plane #02

1. Circle centered at $z_c = -1 + i \sqrt{2}$, of radius $R_c = \pi$.
2. All the points equidistant from 1 and i. Hence: straight line, diagonal through first and third quadrants.
3. Points whose distance to $z = \pm 1$ adds up to the constant 4. Hence: ellipse, with foci $z = \pm 1$, semi-major axis along the real numbers of length $a = 2$, and semi-minor axis along the imaginary numbers of length $b = \sqrt{3}$.

An alternative form to write the set is: $z^2 + 10z + 3 + 3y^2 = 0$ (complete the square) $3(x + \frac{5}{3})^2 + 3y^2 - \frac{16}{3} = 0$ (complete the square) $(x + \frac{5}{3})^2 + y^2 = \frac{16}{3}$. Hence: circle of radius $R_c = 4/3$, centered at $z_c = -5/3$ (center on the negative real axis).

4. $z - \overline{z} = 2i \Im(z)$. Hence: straight line made by all the points with imaginary part 1 (line parallel to the real axis, 1 unit above it).

5. Write $z = x + iy$. Then $z + i\overline{z} = 1 \iff (x+y) + i(y+x) = 1$, which has no solution. Hence: the empty set.

6. Write $z = x + iy$. Then $|z - 1| = 2|z + 1| \iff |z - 1|^2 = 4|z + 1|^2 \iff (x-1)^2 + y^2 = 4((x+1)^2 + y^2) \iff 3x^2 + 10x + 3 + 3y^2 = 0 \iff (x + \frac{5}{3})^2 + y^2 = \frac{16}{3}$. Hence: circle of radius $R_c = 4/3$, centered at $z_c = -5/3$ (center on the negative real axis).

7. Let $w = (z + 1)/(z - 1)$, with inverse $z = (w + 1)/(w - 1)$. Then $\arg(w) = \pm \pi/2 \iff w$ is pure imaginary. Thus write $w = it$, $t$ real, so that $z = (it + 1)/(it - 1) \implies |z| = 1$. On the other hand, if $|z| = 1$ we can write $z = e^{i\theta} \implies w = i \cos(\theta/2)/\sin(\theta/2)$ — thus $w$ is pure imaginary and the condition is satisfied. Hence: the set is the unit circle, with $z = \pm 1$ excluded.

\* The map $z \to w$ is its own inverse. Note that $z = 1 \to w = \infty$ and $z = -1 \to w = 0$, which we must exclude.

2 Sum of exponentials as a ratio of sines

2.1 Statement: Sum of exponentials as a ratio of sines

For any $x$ real, express the finite geometric series

$$S(x) = \sum_{k=-6}^{6} e^{-ikx}$$  \hspace{1cm} (2.1)

as simply a ratio of two sines.

Hint: remember how the formula for the sum of a finite geometric series of real numbers is derived.

What happens with your formula, and its derivation, when $x$ is a multiple of $2\pi$? (Hint: recall L’Hospital rule.)

2.2 Answer: Sum of exponentials as a ratio of sines

Multiply (2.1) by $1 - e^{-ix}$. This yields

$$S(x)(1 - e^{-ix}) = e^{i6x} - e^{-i7x} \iff S(x) = \frac{e^{i6x} - e^{-i7x}}{1 - e^{-ix}}.$$  \hspace{1cm} (2.2)

Now multiply both the numerator and the denominator by $e^{ix/2}$, to obtain

$$S(x) = \frac{\sin(6.5x)}{\sin(0.5x)}$$  \hspace{1cm} (2.3)
Note that this calculation makes no sense when \( 1 - e^{-ix} = 0 \) (i.e.: when \( x \) is a multiple of \( 2\pi \)), because it involves a division by zero. However, (2.3) still makes sense as a limit. If you use L’Hospital rule to calculate its limit as \( x \to 2n\pi \), the answer is

\[
S(2n\pi) = 13. \quad (2.4)
\]

This is the same value (2.1) gives.

### 3 Integer powers and roots #01

#### 3.1 Statement: Integer powers and roots #01

Let \( n > 1 \) be an integer. Then consider the following statements

(a) \((z^{1/n})^n = z\)

(b) \((z^n)^{1/n} = z\),

where \( z = 1 + i(\sqrt{3} - 0.01) \).

(3.1)

Are they correct? Explain why.

#### 3.2 Answer: Integer powers and roots #01

(a) Correct. The \( n \)-th root of a complex number is a multiple valued function. For \( z \neq 0 \) there are \( n \) distinct values that \( z^{1/n} \) can take. However, for every one of them \((z^{1/n})^n = z\).

(b) Incorrect with probability \((n - 1)/n\). The integer power of a complex number is single valued. Hence \( z^n \) is a well defined single number. However, \((z^n)^{1/n}\) can have any of \( n \) distinct values, only one of which is \( z\).

Note: Suppose we make the \( n \)-root single valued by choosing a branch; for example: the principal value.∗ Is then (b) correct? Answer: not necessarily! For example, using principal values: \((-1)^{2^{1/2}} = 1 \neq -1\).

In particular, for \( z \) as in (3.1) and principal value \( n \)-th root, (b) is true for \( n = 2, 3 \); false for \( n > 3 \). Why?

∗ The principal value is defined as follows: write \( z = re^{i\theta} \), with \(-\pi < \theta \leq \pi\). Then \( z^{1/n} = r^{1/n}e^{i\theta/n}\).

### 4 Polynomial equation geometrical solution #02

#### 4.1 Statement: Polynomial equation geometrical solution #02

Consider the polynomial

\[
P(z) = (z - 1)^6 - (z + 1)^6. \quad (4.1)
\]

(a) Show geometrically, without solving for them, that all the roots of \( P(z) \) must lie on the imaginary axis.

Hint: what does \( P(z) = 0 \) say about distances?

(b) Find all five roots of \( P(z) \) algebraically (why only five?).

(c) Challenge: Can you give a geometrical justification for the roots?

Hint: because \( z \) is pure imaginary, \( \overline{z} = -z \). Thus the roots satisfy \((z - 1)^6 - (\overline{z} - 1)^6 = 0\). Given the roots in part b, draw \( z - 1 \) and \( \overline{z} - 1 \), and explain why they are roots with a geometrical argument (based on how powers of a complex number behave).
4.2 Answer: Polynomial equation geometrical solution #02

a. If \( P(z) = 0 \), then \((z - 1)^6 = (z + 1)^6\). Taking absolute values we see that \(|z - 1| = |z + 1|\). Hence \( z \) must be equidistant from 1 and -1. That is, on the imaginary axis.

b. For any root \((z - 1)^6 = (z + 1)^6\) — in particular, \( z \neq \pm 1 \) and we can divide by \((z + 1)^6\). Thus

\[
\left( \frac{z - 1}{z + 1} \right)^6 = 1 \quad \Longleftrightarrow \quad \frac{z - 1}{z + 1} = -e^{-i \theta_n} \quad \text{with} \quad \theta_n = \frac{n \pi}{3},
\]

where \( n \) is an integer taking six consecutive values. However, note that we have to exclude \( \theta_n = \pi + 2m \pi \), because this corresponds to \( z = \infty \). Thus only five roots are left — in terms of (4.1), this is because \( P(z) \) is a 5-th order polynomial; the powers \( z^6 \) cancel out.

From the above is now easy to see that the roots are

\[
z_n = i \tan \left( \frac{1}{2} \theta_n \right), \quad \text{where} \quad \theta_n = 0, \pm \frac{\pi}{3}, \pm \frac{2\pi}{3}.
\]

(4.3)

† We have written the six roots of 1 in a strange way (makes the follow-up algebra easier). Verify this is correct.

c. See figure 4.1.

\[\text{Figure 4.1: Typical root as described by (4.3).}\]

5 Polynomial equation geometrical solution #11

5.1 Statement: Polynomial equation geometrical solution #11

Consider a general polynomial of degree \( n \), with real and positive coefficients

\[P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_2 z^2 + a_1 z + a_0 \quad (a_k > 0 \text{ real}).\]

(5.1)

Show geometrically that there are no roots of \( P(z) \) in the wedge-shaped region of the complex plane defined by

\[|\arg z| < \frac{\pi}{n}.\]

(5.2)

Find an example of a \( P \) as in (5.1) with a root satisfying \(|\arg z| = \pi/n\). Challenge: is there an \( n = 8 \) example? Hint: think how real and imaginary parts relate to the polar representation in the plane.
5.2 Answer: Polynomial equation geometrical solution #11

Consider a \( z \) in the region defined by (5.2). Then \( z = r e^{i \theta} \) (with \(-\pi/n < \theta < \pi/n\) and \( r = |z| \)) and \( z^k = r^k e^{i \phi} \), where \( \phi = k \theta \). The following cases then apply.

1. **z is real and positive**: i.e.: \( \theta = 0 \). Then \( P(z) > 0 \), so that \( z \) is not a root.

2. \( \text{Im}(z) > 0 \); i.e.: \( 0 < \theta < \pi/n \). Then \( \text{Im}(z^k) > 0 \) (\( 1 \leq k \leq n \)) and \( \text{Im}(P(z)) = a_n \text{Im} z^n + \ldots + a_1 \text{Im} > 0 \). Hence \( z \) is not a root.

3. **\( \text{Im}(z) < 0 \); i.e.: \(-\pi/n < \theta < 0 \).** An argument similar to the one in item 2 then yields \( \text{Im}(P(z)) < 0 \).

**Example**: Let \( P(z) = z + 1 \). Then \( n = 1 \), \( z = -1 \) is a root (only one), and \( \arg z = \pi/n \).

However, **no example with \( n > 1 \) is possible**.

**Proof**. Let \( \arg z = \pi/n \). Then, in the argument above, either item 2 or item 3 applies, except that: (i) In item 2, \( \text{Im}(z^k) > 0 \) for \( 1 \leq k < n \) (not \( 1 \leq k \leq n \)). However, it is still true that \( \text{Im}(P(z)) > 0 \). (ii) A similar change happens for item 3.

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6 Roots of polynomials #01

6.1 Statement: Roots of polynomials #01

Find all the roots of the following polynomial equations:

- (1) \( z^2 + z - 2 = 0 \).
- (2) \( z^2 - (2 - i) z - 2 i = 0 \).
- (3) \( z^3 + i z^2 - z - i = 0 \).
- (4) \( z^4 - 2 z^2 - i = 0 \).

**Hint**: for (3) you can easily guess (at least) a root. Then factor the root out.

6.2 Answer: Roots of polynomials #01

(1) \( 0 = z^2 + z - 2 = (z + 1/2)^2 - 9/4 \). Thus \( z = -1/2 \pm 3/2 \). That is \( z = 1 \) and \( z = -2 \).

(2) \( 0 = z^2 - (2 - i) z - 2 i = (z - 2 + i)/2 - 3+4i/2 \). Hence: the roots are \( z = 2 - i/2 \) \( \pm \sqrt{3 + 4i}/2 \).

(3) \( 0 = z^3 + i z^2 - z - i = (z^2 - 1) + i (z^2 - 1) \). Hence: the roots are \( z = \pm 1 \) and \( z = -i \).

(4) \( 0 = z^4 - 2 z^2 - i = (z^2 - 1)^2 - 1 - i \), so that \( z^2 = 1 \pm \sqrt{1 + i} \). Hence: the roots are \( z = \pm \sqrt{1 \pm \sqrt{1 + i}} \).

Note \( 1+i = \sqrt{2} e^{i \pi/4} \). Thus \( \sqrt{1+i} = \sqrt{2} e^{i \pi/8} \) and \( -\sqrt{1+i} = \sqrt{2} e^{-i \pi/8} \), using the principal branch definition.

† Here we **define complex number using the principal branch**, which yields a single valued function on the complex plane (discontinuous along the negative real semi-axis). As explained in the lectures, because the square root is multiple valued, you cannot just write \( \sqrt{z} \) without specifying which root.

Recall that the principal value is defined by: Let \( z = r e^{i \theta} \), with \(-\pi < \theta \leq \pi \) (principal value of the argument). Then \( \sqrt{z} = \sqrt{r} e^{i \theta/2} \) — here \( \sqrt{r} \) is the usual non-negative real root of a non-negative real number.

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THE END.