Taylor and Laurent Series

Examples

Ex 1 Example 1

Taylor centered at \( z_0 \) for a polynomial

Let \( P(z) = \sum_{n=0}^{N} a_n z^n \) be a polynomial

\( (a_n \text{ some complex const.}) \)

Then

\[
P(z) = \sum_{n=0}^{N} a_n (z_0 + (z - z_0))^n = \sum_{n=0}^{N} a_n \sum_{l=0}^{n} \binom{n}{l} z_0^n (z - z_0)^l = \sum_{l=0}^{N} \left( \sum_{n=l}^{N} a_n \binom{n}{l} z_0^{n-l} \right) (z - z_0)^l \]

\[
\hat{a}_l = \sum_{l=0}^{N} \hat{a}_l (z - z_0)^l
\]

where \( \binom{n}{l} = \frac{n!}{l!(n-l)!} \)

\[
\hat{a}_l = l! P^{(l)}(z_0) \quad (2)
\]

Intuition: Taylor's theorem says that analytic functions behave like infinite degree polynomials where they are analytic.
Example 2: Let \( P(z) = \prod_{k=1}^{N} (z-z_k) \) be a polynomial with \( N \) distinct roots \( \{z_k\}_{k=1}^{N} \). Then we can write, using the "cover up" rule,

\[
\frac{1}{f(z)} = \sum_{k=1}^{N} \frac{A_k}{z-z_k}, \quad \text{where} \quad A_k = \prod_{p \neq k} (z_k-z_p)
\]

Now let \( \Gamma_k = |z_k| \), where we order the \( \{z_k\} \) so that \( \Gamma_1 \leq \Gamma_2 \leq \ldots \leq \Gamma_N \).

Then \( f \) is analytic in each of the regions: \( \Gamma < \Gamma_1; \quad \Gamma_1 < \Gamma < \Gamma_{k+1}; \quad \Gamma_N < \Gamma \), where \( k = 1; \ldots; N-1 \).

Note: If \( \Gamma_k = 0 \) then \( R_k \) is empty.

If \( \Gamma_k = \Gamma_{k+1} \) then \( R_k \) is empty.

Furthermore,

\[
\frac{1}{z-z_k} = \sum_{l=1}^{\infty} \frac{1}{z_n} \quad \text{for} \quad |z| > \Gamma_k
\]

For simplicity, normalize \( P \) so that \( P(0) = 0 \).

For \( |z| \) large, we prove this rule later. \( \{ \text{Ex 3 and Ex 4} \} \)
and \( \frac{1}{Z-Z_0} = - \sum_{n=0}^{\infty} \frac{1}{Z_0^{n+1}} Z^n \) \( \text{for } |Z| < \Gamma_0 \)

Thus:

\[
\text{Taylor series for } Z \text{ in } \mathbb{R}_+^d \quad f = \sum_{n=0}^{\infty} \alpha_n Z^n
\]

where \( \alpha_n = - \sum_{l=1}^{N} A_l Z_l^{-n-1} \)

Note that this requires \( \Gamma_j > 0 \) \( \Rightarrow \) no \( Z_j \) vanishes.

\[
\text{Laurent series for } Z \text{ in } \mathbb{R}_-^d \quad f = \sum_{n=1}^{\infty} \beta_n Z^{-n} + \sum_{n=0}^{\infty} \alpha_n Z^n
\]

where \( \beta_n = \sum_{j=1}^{l} A_j Z_j^{-n-1} \) and \( \alpha_n = \sum_{j=l+1}^{N} A_j Z_j^{-n-1} \)

Note that this requires \( \Gamma_j < \Gamma_{j+1} \) \( \Rightarrow \) \( |Z_j| > 0 \) for \( j > l \) and \( \alpha_n \) is well defined.

\[
\text{Laurent series for } Z_1 \text{ in } \mathbb{R}^-_1 \quad f = \sum_{n=1}^{\infty} \beta_n Z^{-n} \quad \text{with } \beta_n = \sum_{l=1}^{N} A_l Z_l^{-n-1}
\]

Intuition: To produce (7-8) we have only
properties of the geometric series. The
Laurent series theorem says that formulas
like these are true in general, not just
for rational functions. 

For a rational function, $Q(z)/P(z)$ with $Q$
and $P$ polynomials, first expand as above
$1/P$ and then multiply by $Q$
(without common roots).

Note: The case when $P$ has multiple roots is
technically more cumbersome, but can be
done as well. For this we need a more sophisticated
version of the cover up rule, and the expansions
for $(z-z_\alpha)^P$ (which follow by taking derivatives
in (6)). We will not do this here.

Ex 3 Example 3 (Algebraic proof of the cover up
rule). Let $P(z) = \alpha \prod_{1}^{N} (z-z_\alpha)$ be a
polynomial with $N$ distinct roots $\{z_\alpha\}_{\alpha=1}^{N}$
and let \( Q = Q(z) \) be a polynomial of degree less than \( N \). Then (note: \( a \neq 0 \) in a const.)

\[
f(z) = \frac{Q(z)}{P(z)} = \sum_{\ell=1}^{N} \frac{A_\ell}{(z-z_\ell)}
\]

where \( A_\ell = \frac{Q(z_\ell)}{(a \prod (z_\ell - z_p))} \).

\[\text{Proof: Multiply both sides of (10) by } P(z). \text{ Then (10) becomes}
Q(z) = \sum_{\ell} Q(z_\ell) \prod_{p \neq \ell} \frac{z-z_p}{z_\ell - z_p} \]

Now:

\[R(z) = Q(z) - g(z) \text{ is a polynomial of degree less than } N, \text{ which satisfies } R(z_\ell) = 0 \]

for \( \ell = 1, \ldots, N \). Thus it must be \( R \equiv 0 \). QED

**Ex 9**  
Complex variable proof of the cover up rule.

**Proof:** Consider the Laurent expansion for \( f(z) \) valid for \( 0 < |z-z_\ell| < \text{some } R \). The singular part for this expansion is \( \frac{A_\ell}{z-z_\ell} \).
\[
\frac{Q(z)}{P(z)} = \sum_{i=1}^{N} \frac{A_i}{z-Z_i} = g(z) \text{ is not singular at } z = Z_i \text{ (removed the singularity).}
\]

Hence \( g(z) \) is \underline{entire}, and vanishes as \( |z| \to \infty \). Thus (Liouville's Theorem) \( g \equiv 0 \). \( \text{QED} \)

This proof tells us what to do in the general case (multiple roots): at each root \( Z_i \), pick up the singular part of the corresponding Laurent expansion! Furthermore: if \( \text{degree } g \geq \text{degree } P \), add the behavior at \( \infty \) (the regular part of the Laurent expansion for \( |z| > R = \max |Z_i| \)).

\underline{Example 5} Laurent expansions for \( \frac{1}{\sin z} \) near the roots of \( \sin z \).

Since \( \sin z = (-1)^n \sin (z - n\pi) \), we can write

\[
\sin z = (-1)^n \sum_{l=0}^{\infty} \frac{(-1)^l (z-n\pi)}{(2l+1)!} = (-1)^n (z-n\pi) (1 - g(z)),
\]
where \( g(z) = \sum_{l=1}^{\infty} (-1)^{l+1} \frac{(z-n\pi)^l}{(2l+1)!} \).

\[ g(z) = \frac{1}{6}(z-n\pi)^2 - \frac{1}{120} (z-n\pi)^4 + \ldots \]

which is entire (Note also that \( g \) is small for \( z \) close to \( n\pi \)). Then:

\[
\frac{1}{\sin z} = \frac{(-1)^n}{z-n\pi} \frac{1}{1-g} = \frac{(-1)^n}{z-n\pi} \sum_{l=0}^{\infty} g^l
\]

\[ = \frac{(-1)^n}{z-n\pi} \left\{ 1 + \frac{1}{6} (z-n\pi)^2 + \frac{7}{360} (z-n\pi)^4 + \ldots \right\} \]

where the last line follows from substituting (14) into \( \Sigma \), and collecting equal powers of \( (z-n\pi) \). We can do this because (14) and \( \Sigma \) both converge absolutely for \( |z-n\pi| \) small enough to guarantee \( |g| < 1 \). However, once we do this, the Laurent expansion theorem tells us that

\[ (15) \text{ converges for } 0 < |z-n\pi| < 1. \]
and we can "forget" the intermediate step where \( |g| < 1 \) is needed!

Example 6

**Expansion of \( \frac{1}{\sin z} \) in fractions**

Following the proof in example 4, subtract from \( \frac{1}{\sin z} \) the singular part at each root --- combining the singular parts for \( \pm n\pi \) into 1.

Then we can see that

\[
\frac{1}{\sin z} = \frac{1}{Z} + \sum_{n=1}^{\infty} \frac{(-1)^n 2Z}{z^2 - n^2\pi^2}
\]

Why? Because the expression on the right defines a function analytic in the complex plane, minus the points \( z = n\pi \) (\( n \) integer) --- see notes Ex6 below. Furthermore:

\[
\frac{(-1)^m}{z - n\pi} + \frac{(-1)^m}{z + n\pi} = (-1)^m \frac{2Z}{z^2 - n^2\pi^2}; \ n \neq 0.
\]

Why? This is so the series in (17) converges!
the difference between the left and right sides of (17) is, by construction entire and bounded (see notes Ex.6 below). Hence the difference is a constant. Take then 
\[ z = iy, \quad y > 0, \] and let \( y \to \infty \). Since both sides vanish, the constant is zero. QED

**Notes Ex.6** The argument above involves a couple of steps that require some "sweat" and the theorem (analytic)

If a sequence of analytic functions (in some open region \( R \)) \( g_n(z) \) converges uniformly to a function \( f(z) \) in \( R \), then \( f \) is analytic.

Uniformly here means that \( |f(z) - g_n(z)| \leq c_n \) for all \( z \in R \), where \( c_n \) is a constant, and \( c_n \to 0 \) as \( n \to \infty \). The proof is actually not hard (it involves showing that...
f has the "path independent" property) but I will skip it here. Then

(ii) Use (18) to show the r.h.s. in (17) is analytic in any region |z| < R and

|z − nπ| > ε (0 < ε << 1 << R) where the

series converges uniformly & absolutely

[Bound |zz/(z2−nπ)| there]. Skip details.

(ii) Showing that the difference between the

left and right side of (17) is also a bit

technical and we skip it. The idea is to

use periodicity. Then it is enough to find

a bound for |Re(z)| ≤ π/2, which is not

too hard. Skip details.

Ex7 Example 7 The function \[ f(z) = \frac{\cos z}{\sin z} \]

Just as in Ex 5-7, it is easy to see that

the singular part of \( f(z) \) at \( z = n\pi \)

is \( 1/z - n\pi \) (see notes Ex7), from
which we conclude \[
\frac{\cos z}{\sin z} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{z^{2n}}{z^2 - n^2 \pi^2}.
\]

**Notes Ex7:** The Taylor expansion for \( \sin z \) is given in (13). On the other hand \( \cos z = (-1)^n \left\{ 1 - \frac{(z-n\pi)^2}{2} + \ldots \right\} \). From this the singular part for \( \cos z/\sin z \) follows.

**Remark** Examples 6 and 7 show that the "cover up" rule (Ex 3-4) also applies to (some) analytic functions (trig. at least). Next we show that polynomial factorization generalizes as well.

**Ex 8** **Example 8:** Express \( \sin z \) as a product.

Note that \( \frac{\cos z}{\sin z} = \frac{d}{dz} \log \sin z \).

Thus, using (18)
\[
\frac{d}{dz} \log \sin z = \frac{d}{dz} \left\{ \log z + \sum_{n=1}^{\infty} \log \left(1 - \frac{z^2}{n^2 \pi^2}\right) \right\}
\]

+ Which branch of the log we use does not matter because they differ by a constant, which \( \frac{d}{dz} \) kills.
Note: when integrating \( \frac{2z}{z^2 - n^2 \pi^2} \), I selected the constant of integration so that the resulting series in (19) is absolutely convergent. This then justifies the term by term integration.

\[ \text{uniformly for } |z| < r < \pi \]

We can now rewrite (19) in the form

\[ \frac{d}{dz} \log \sin z = \frac{d}{dz} \log \left\{ z \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{n^2 \pi^2} \right) \right\} \]

\[ \Rightarrow \log \sin z = \log \left\{ z \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{n^2 \pi^2} \right) \right\} + \text{const} \]

\[ \Rightarrow \sin z = \text{const.} \cdot z \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{n^2 \pi^2} \right)^{1/n} \cdot g(z) \]

We can get the constant by evaluating the derivative of both sides at \( z = 0 \)

\[ 1 = \cos 0 = \text{const.} \cdot \left\{ z g'(z) + g(z) \right\} \]

but \( g(0) = 1 \). Thus

\[ \sin z = z \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{n^2 \pi^2} \right) \]

Note: as usual I have skipped tedious analysis.
Ex 9 Example 9 The "analog" of the theorem in (18) (pape TL09) is false for functions of a real value.
Consider the function
\[ f(x) = \sum_{n=0}^{\infty} a^n e^{i b^n x}, \quad x \text{ real}, \]
where \(0 < a < 1\) and \(b\) is a positive odd integer such that \(ab > 1 + \frac{3}{2} \pi\).

The series in (21) is absolutely and uniformly convergent; hence \( f \) is continuous.

However: \( f \) is no-where differentiable.

This even though each of the terms in the series is smooth.

Note Ex9.1: The graph of \( f \) is fractal, with jagged peaks everywhere.

Note Ex9.2: The real part of (21) is called the Weierstrass function, who proved the result

\( \text{I am using here a result from 18.100} \)
in the late 19th century. The proof uses calculus only, but it is a bit long and involved (so we skip it here). Later Hardy showed that \( ab > 1 + \frac{3}{2} \pi \) is not needed, and that \( ab \geq 1 \) is enough — in fact, \( b \) does not have to be integer either.

**Note Ex 3.3** But, you may say here: the functions in (21), \( e^{ib^nx} \), are actually analytic! How does (22) not contradict (18)? Should not \( f \) be analytic?

OK, so let us consider \( f(z) = \sum_{n=0}^{\infty} a^n e^{ib^nz} \) (23)

... this converges, as required by (18) for \( \text{Re}(z) > 0 \), and indeed (23) is analytic for \( \text{Re}(z) > 0 \), but the real axis is on the boundary, and (18) says nothing about the boundary!

In the next example we develop this in more detail.
Ex10 Example 10 "Natural Boundaries"

The Taylor series theorem says that for an analytic function \( f = f(z) \) in some region \( A \), the Taylor series
\[
\sum_{n=0}^{\infty} a_n (z - z_0)^n
\]
converges to \( f \) for \( |z - z_0| < R \), where \( R \) is the radius of the largest disc centered at \( z_0 \) and included in \( A \).

In the examples we have seen so far, \( f \) typically has a few singularities along \( |z - z_0| = R \), but other than that it makes sense beyond \( z - z_0 \). For example

(a) Look at Ex2 and place all the \( z_0 \) in the unit circle.

(ii) \( f(z) = \frac{1}{\sin W} \), where \( W = \frac{1 - z}{1 + z} \). (24)
You can check that this has singularities on

$$z = \frac{1+i \pi n}{1-i \pi n}$$

$$n = 0, \pm 1, \pm 2, \ldots$$

All in the unit circle, with an accumulation point at $$z = -1$$. Still, the function is well defined beyond $$|z| = 1$$. However, the example in (2.3) should give you pause. This function is analytic for $$\text{Re}(z) > 0$$, but it is stopped "cold" by the real axis.

This is called a "Natural Boundary".

Let us look at another example:

$$f(z) = \sum_{n=0}^{\infty} z^n$$

This is a Taylor series, that converges for $$|z| < 1$$ and defines $$f$$ as an analytic function in the unit disk. However, now note:
(1) For \( z = r, 0 \leq r < 1 \), \( f(z) = \sum_{n=0}^{\infty} r^{2n} \to \infty \) as \( r \to 1 \)

(2) \( f(z) = z + f(z^2) \)

\[ \therefore \quad f(-r) \to \infty \quad \text{as} \quad r \to 1 \]

\[ \therefore \quad f(\pm ir) \to \infty \quad \text{as} \quad r \to 1 \]

And so on; in fact

for \( z = r z_0, z_0^{2n} = 1 \), \( f(z) \to \infty \) as \( r \to 1 \)

Thus \( f \) has a dense set of singularities on \( |z| = 1 \)

Again \( |z| = 1 \) is a natural boundary.

For a final example, consider \( f(z) = \sum_{n=0}^{\infty} z^n \! ! \)

In this case one can show

\[ f(z) \to \infty \quad \text{as} \quad r \to 1 \quad \text{for} \]

\[ z = r \exp \left\{ i \frac{P \pi}{9} \right\}, \enspace P \text{ integer} \]

The reason is that for \( n \geq q + 2 \), \( z^n = r^n \).
Real part of the Weierstrass function; with terms added from $n=0$ to $n=N-1$. 

Weierstrass: $(a, b) = (0.5, 13)$; terms $N = 2$. 

![Graph of Weierstrass function](image)
The plot is fractal. If one blows up a small piece, it looks like the entire plot. In the numerics, of course, we are limited by the resolution (N), not to mention the pixel size in the figure.

Weierstrass: \((a, b) = (0.5, 13); \) terms \(N = 3.\)
Weierstrass: \((a, b) = (0.5, 13)\); terms \(N = 5\).
Weierstrass: \((a, b) = (0.5, 13);\) terms \(N = 10.\)