Taylor Series  Cover Ch.9 §9.1 to §9.4 notes.

Remark to Theorem 9.1 p.103

A series is said to be Absolutely Convergent if \[ \sum_{n=0}^{\infty} |a_n| < \infty \,.

Absolutely convergent series have nice properties...

...basically they behave the same way as finite sums:

(i) You can add them in any order

(ii) You can multiply or add two (or more) AC series term by term.

(iii) If you have a double summation you can commute the indexes

\[ \sum_{n} \sum_{m} b_{nm} = \sum_{m} \sum_{n} b_{nm} \]
etc.

By contrast, consider the example

\( \Sigma (s)^n / n \)

If you change the order of summation you can get this series to converge to any number in \([-\infty, \infty]\) or not converge and oscillate.

Remark to example 9.7 p 106

Consider the entire function \( f(z) = e^{z+a} \) with a some arbitrary constant. Then

\[ f^{(n)}(z) = e^{z+a} \implies f(z) = \sum_{n=0}^{\infty} \frac{z^n e^a}{n!} = e^z e^a \]

This is yet another proof that

\( e^{z_1 + z_2} = e^{z_1} e^{z_2} \)
Remark to example 9.10 p107

This example gives the impression that you need a clever trick to do it.

Not so. "Brute force" works. Just more work

\[ f(z) = \frac{1+2z^2}{z^3+z^5} = \frac{1}{z^3} \left( 1+2z^2 \right) \frac{1}{1+z^2} \]

\[ = \frac{1}{z^3} \left( 1+2z^2 \right) \sum_{0}^{\infty} (-z^2)^n = \]

\[ = \frac{1}{z^3} \left\{ \sum_{0}^{\infty} (-z^2)^n + 2 \sum_{1}^{\infty} (-z^2)^n \right\} \]

\[ = \frac{1}{z^3} \left\{ 1 - \sum_{1}^{\infty} (-z^2)^n \right\} \]

\[ = \frac{1}{z^3} \left\{ 1 + z^2 - \sum_{0}^{\infty} (-z^2)^{n+2} \right\} \]

\[ = \frac{1}{z^3} + \frac{1}{z} - \sum_{0}^{\infty} (-1)^n z^{2n+1} \checkmark \]

Example 9.11 p108 \Rightarrow \mathbf{f}(0) = n! \left( 1 + \frac{1}{2!} + \ldots + \frac{1}{n!} \right)
Example: Taylor series for \( f(z) = \sqrt{1+z} \)

Note: \( f'(z) = \frac{1}{2\sqrt{1+z}} \); \( f(0) = \sqrt{1} \); \( f > 0 \)

\[ f''(z) = -\frac{1}{4(1+z)^{3/2}} \]
\[ f'''(z) = \frac{3}{8(1+z)^{5/2}} \]
\[ f^{(iv)}(z) = -\frac{5\times3}{16(1+z)^{7/2}} \]
\[ f^{(n)}(z) = \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) \cdots \left(\frac{1}{2}-n+1\right) \frac{1}{(1+z)^{1/2-n}} \]

Use induction for now.

\[ f(z) = \sum_{n=0}^{\infty} \frac{(1/2)!}{(1/2-n)!} z^n \quad |z| < 1 \]

Same as \( (1+z)^m = \sum_{n=0}^{m} \binom{m}{n} z^n \quad m \text{ integer!} \)

\[ \binom{m}{n} = \frac{m!}{n!(m-n)!} \]