Finish Ch.7  Do examples 7.6 & 7.7 with notes.

Start Ch.8  2-D hydrodynamics and complex potentials

Example applications (stuff here translates to other areas)

Later on I will tell you about Stokes flow and the bi-harmonic equation. The stuff in Ch.8 is for inviscid flow. Stokes is the other end.

Why divergence free? \[(8.3.1)\text{ in notes}\]

In general, let \( p = p(x,t) \) be the mass density (mass per unit volume) and \( \bar{u} = \bar{u}(x,t) \) the flow velocity

(Note: here dimension = 2 or 3, dust matter)
Consider now some fixed volume $\Omega$ with boundary $\partial \Omega$ (a potato and its skin). At every point along $\partial \Omega$ let $\hat{n}$ be the unit outside normal.

Now the mass in $\Omega$ is

$$ M = \int_{\Omega} \rho \, dV \quad (1) $$

and the flow of mass out of $\Omega$ is

$$ M_f = \int_{\partial \Omega} \rho \hat{u} \cdot \hat{n} \, dA \quad (2) $$

Since mass is conserved

$$ \frac{dM}{dt} = M_f \quad (3) $$

Further Gauss Theorem says

$$ M_f = \int_{\Omega} \text{div}(\rho \hat{u}) \, dV \quad (4) $$

also

$$ \frac{dH}{dt} = \int_{\Omega} p_t \, dV \quad (5) $$
\[
\text{(Note: here we assume } p \text{ and } \mathbf{u}\ 	ext{ are smooth enough)}
\]

\[\text{The \,(3-5) \Rightarrow } \int_{\Omega} (p_t + \text{div}(\rho \mathbf{u})) \, dV = 0 \quad (6)\]

But this has to be true for every volume \(\Omega\) :

\[\frac{\rho_t + \text{div}(\rho \mathbf{u}) = 0}{\text{Conservation of Mass}}\]

This is true both for gases and liquids. However, for liquids it is very hard to change \(\rho\) : approximate \(\rho = \text{const.}\)

\[\Rightarrow \quad \text{div} \mathbf{u} = 0 \quad (8)\]

\(\rho \approx 1\text{ g/cm}^3\) for water.

\text{Why in rotational? : this is an assumption!}

For some flows it is a good approx., for others it is not.
Easy way to see eddy flow not rotational

\[ \mathbf{F} = \left( -\frac{y}{r^2}, \frac{x}{r^2} \right) \]

Let \( \rho = \frac{1}{2} \ln(x^2 + y^2) \)

then \( u = -\rho_y, \quad v = \rho_x \)

\[ \therefore \ \text{curl} \mathbf{F} = \nabla \times \mathbf{F} = \nabla u = \Delta \rho = 0 \ \text{why} \ 0 \ ?? \]

Also \( \text{div} \mathbf{F} = \nabla \cdot \mathbf{F} = u_x + v_y = -\rho_y x + \rho_x y = 0 \) \( \text{(9)} \)

I plan to cover up to including § 8.4

To follow the arguments in (§ 8.4) convenient to have

Cauddy-Riemann \( f = u + iv : \)

\[ \begin{align*}
  u_x &= i v_y \\
  u_y &= -i v_x
\end{align*} \]

(10)

\( f = \phi + i \psi : \)

\[ \begin{align*}
  \phi_x &= \psi_y \\
  \phi_y &= -\psi_x
\end{align*} \]

(11)

Note then \( f^1 = u_x + i v_x = u_x - i u_y = \phi_x - i \phi_y \) \( \text{(12)} \)

Note that (33) p. 83 \( \{ \text{io (10) with} \ v \to -v \}

\[ \text{end of § 8.3.3} \] \( \text{i.e.} \ f = u - i v \)
Examples for Complex potentials

Example 1

\[ \Phi = \phi + i \psi = \frac{\partial \log z}{\partial \log z} = \ln r + i \theta \]

Thus \[ \phi = \ln r = \ln \sqrt{x^2 + y^2} = \frac{1}{2} \ln (x^2 + y^2) \]

and \[ u = \phi_x = \frac{x}{r^2}; \quad v = \phi_y = \frac{y}{r^2} \]

This is the source field.

Example 2

\[ \Phi = \phi + i \psi = -i \log z = \Theta - i \ln r \]

Thus \[ \phi = \Theta \] so that

\[ u = \Theta_x = (-\ln r)_y = -y/r^2 \] \quad Eddy (vortex) field

\[ v = \Theta_y = (-\ln r)_x = x/r^2 \]

Use Cauchy-Riemann here to make computing \( \Theta_x \) and \( \Theta_y \) easier!