Harmonic Functions

The lecture will track Ch. 7 pp 81-83 of the course web notes, with the changes and additions below.

In a prior lecture I went through many of the examples of applications where harmonic functions occur. We will go into some of them in some detail later. The point to be made now is that the Laplace operator

$$\Delta = \nabla^2 = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2} \quad d = \text{dimension} \quad (1)$$

is ubiquitous in science and engineering.

Here we will study the case $d = 2$. 
where \[ \Delta = \nabla^2 = \partial_x^2 + \partial_y^2 \] (2)

Note the symbol \( \Delta \) (Delta) which is widespread notation for the Laplacian.

From notes:

- Definition of harmonic 7.1
- Notation 7.2

\{ Theorem 7.2 \}
\{ Theorem 7.3 \}

Analytic functions and harmonic 7.2.1

Note here harmonic means \( \Delta u = 0 \) \( \uparrow \)
with \( u \) twice continuously differentiable \( \downarrow \)

Note to proof of theorem 7.3

In prior lectures we saw that analytic and path independent are equivalent; and thus
\[ f(z) = \int_{Z_0}^{z} g(z) \, dz + \text{const.} \]
is an anti-derivative of \( g \), where the integration path (on a simply
connected region) does not matter.

Note to theorem 7.3: \( v \) is called the harmonic conjugate to \( u \). This theorem says that any harmonic function has an harmonic conjugate on a simply connected region.

If \( u \) is harmonic and 
\[
\begin{align*}
  u_y &= u_x \\
  v_x &= -v_y
\end{align*}
\]  

(4)

\( v \) is called an harmonic conjugate of \( u \).

\[ \left\{ \text{Note that (4) are just the Cauchy-Riemann Conditions} \right\} \]

On a simply connected region, if \( v_1 \) and \( v_2 \) are harmonic conjugates of \( u \), then 
\[
|v_1 - v_2 = \text{const.}| 
\]  

(5)

Why? Because \( \nabla v_1 = \nabla v_2 \)
A second proof that the real and imaginary parts of an analytic function are harmonic

Note that this proof uses the same idea we used before to prove

\[ \text{Analytic} \rightarrow \infty \text{ derivatives} \]

We start from Cauchy

\[ f(z) = \frac{1}{2\pi i} \oint \frac{f(w) \, dw}{w-z} \]

and fix the contour. Then

consider what happens inside the contour

Then

a) Because \( \frac{1}{w-z} \) has infinite derivatives with respect to \( z \), and the contour is fixed, we can switch integration and derivation

\[ \rightarrow \text{Cauchy's formula for derivatives} \]

b) Because \( \frac{1}{w-z} \) is harmonic, and the

\[ \text{Because any point can be included inside some contour, this applies everywhere.} \]
contour is fixed, again we can switch derivation and integration

\[ \Delta \frac{1}{\omega - z} = 0 \]  

(7)

In formulas

\[ \Delta \left( \frac{1}{\omega - z} \right) = 0 \]  

(8)

where \( \omega \) is fixed, \( z = x + iy \), and \( \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \)

Thus

\[ \Delta f = \frac{1}{2\pi i} \oint \Delta \left( \frac{1}{\omega - z} \right) f(\omega) \, d\omega = 0 \]

\[ \Delta f = 0 \]  

(9)

But \( f = u + iv \), and \( \Delta \) is a real operator:

\[ 0 = \Delta u + i \Delta v \]

\[ \Rightarrow \Delta u = \Delta v = 0 \]  

(10)

**Warning:** The way the notes are written makes it look as if you can prove this using \( \text{Re} \left( \frac{1}{z} \right) \) for \( u \) & \( \text{Im} \left( \frac{1}{z} \right) \) for \( v \). This is not
true! Because \( f(w)dw \) is complex, \( u \) is a linear combination of the real & imaginary parts of \( \frac{1}{w-z} \).

I will aim to correct this in the notes.

7.4 p85 Maximum principle and mean value property

Direct proof of the mean value property

Let \( J(r) = \frac{1}{2\pi} \int_0^{2\pi} u(x_o + r \cos \theta, y_o + r \sin \theta) \, d\theta \)

\( u \) harmonic \( \quad \Box \) \( \tag{11} \)

Then

\[
\frac{d}{dr} J = \frac{1}{2\pi} \int_0^{2\pi} (u_x \cos \theta + u_y \sin \theta) \, d\theta = \left[ \text{Gauss} \right] \\
= \frac{1}{2\pi} \int_{\text{Disk}} (u_{xx} + u_{yy}) \, dA = 0 \quad \Box \tag{12}
\]

Thus \( J = \text{constant} \) and \( \lim_{r \to 0} J = u(x_o, y_o). \quad \Box.E.D. \tag{13} \)
This proof shows that the mean
value theorem is not true just in 2-D,
but in any dimension

On the other hand, results involving
the harmonic conjugate \( v \) are 2-D

They have no analog in 3-D.

7.5 p 85 Orthogonality of level curves

Recall that for \( u = u(x,y) \) with \( \nabla u \neq 0 \)
then \( \nabla u \) perpendicular to
\( u = \text{constant} \) \( \tag{14} \)

For any vector \( \mathbf{p} = (p_1, p_2) \) let
\( \mathbf{p}^\perp = (-p_2, p_1) \) rotation by 90°
Now note for \( u \) and \( v \) with
\[
\nabla u \neq 0, \quad \nabla v \neq 0
\]
\[\nabla u \perp \nabla v \quad \iff \quad \text{Cauchy-Riemann} \tag{15}\]

Thus Cauchy-Riemann says at any point where \( f'(z) \neq 0 \) the level curves for \( u = \Re(f) \) \tag{16}
is some perpendicular to the one for \( v = \Im(f) \)

Why do we need \( f' \neq 0 \iff \nabla u \neq 0 \iff \nabla v \neq 0 \)?

Because otherwise the level curves need not be smooth and may not have a well defined tangent and normal.

We will see this in the examples.