#1a $f(\theta)=\theta,\ 0<\theta<\pi \qquad f(\theta)=a_0+\sum_{n=1} A_n\cos(n\theta)$

$$\int_0^\pi f(\theta)\cos(n\theta)d\theta=\int_0^\pi a_0\cos(n\theta)+a_1\cos(\theta)\cos(n\theta)+a_2\cos(2\theta)\cos(n\theta)+\dots d\theta$$

$$\int_0^\pi f(\theta)\cos(n\theta)d\theta = a_n\int_0^\pi \cos^2(n\theta)d\theta = a_n\frac{\pi}{2} \qquad \boxed{a_n=\frac{2}{\pi}\int_0^\pi f(\theta)\cos(n\theta)d\theta}$$ ✓

#1b Set $\theta=0$, prove $\frac{\pi^2}{8}=1+\frac{1}{3^2}+\frac{1}{5^2}+\dots$

$$f(0)=a_0+\sum_{n=1}^{\infty} A_n\cos(n(0))=\frac{2}{\pi}\int_0^\pi f(\theta)d\theta+\frac{2}{\pi}\cos(\theta)\int_0^\pi f(\theta)\cos(\theta)d\theta$$

$$+\frac{2}{\pi}\cos(2\theta)\int_0^\pi f(\theta)\cos(2\theta)d\theta+\dots$$

$$=\frac{2}{\pi}\left(\frac{\pi}{2}-\frac{4}{\pi}+0-\frac{4}{9\pi}+0-\frac{4}{25\pi}+\dots\right)$$

$$=1-\frac{8}{\pi^2}-\frac{8}{9\pi^2}-\frac{8}{25\pi^2}-\dots \rightarrow 0=1-\frac{8}{\pi^2}\left(1+\frac{1}{9}+\frac{1}{25}+\dots\right)\rightarrow \boxed{\frac{\pi^2}{8}=1+\frac{1}{3^2}+\frac{1}{5^2}+\dots}$$ ✓

#1c $\frac{1}{\pi}\int_0^\pi f^2(\theta)d\theta=a_0^2+\frac{1}{2}\sum_{n=1}^{\infty}A_n^2$, prove $\frac{\pi^4}{96}=1+\frac{1}{3^4}+\frac{1}{3^5}+\dots$

$$\frac{1}{\pi}\int_0^\pi \theta^2 d\theta=\left(\frac{2}{\pi}\right)^2\left(\int_0^\pi\theta d\theta\right)^2+\frac{1}{2}\left(\frac{2}{\pi}\right)^2\left[\left(\int_0^\pi f(\theta)\cos\theta d\theta\right)^2+\left(\int_0^\pi f(\theta)\cos2\theta d\theta\right)^2+\dots\right]$$

$$\frac{1}{\pi}\left(\frac{\theta^3}{3}\right)\Big|_0^\pi=\frac{4}{\pi^2}\left(\frac{\pi}{2}\right)^2+\frac{2}{\pi^2}\left[(-2)^2+0+\left(\frac{-2}{9}\right)^2+0+\left(\frac{-2}{25}\right)^2+\dots\right]$$

$$\frac{\pi^2}{12}=\frac{4}{\pi^2}\left(\frac{\pi^2}{4}\right)+\frac{2}{\pi^2}\left(4+\frac{4}{3^4}+\frac{4}{5^4}+\dots\right)$$

$$\frac{\pi^2}{12}=\frac{8}{\pi^2}+\frac{8}{3^4\pi^2}+\frac{8}{5^4\pi^2}+\dots$$

$$\frac{\pi^4}{12}=8+\frac{8}{3^4}+\frac{8}{5^2}+\dots \qquad \boxed{\frac{\pi^4}{96}=1+\frac{1}{3^4}+\frac{1}{5^4}+\dots}$$ ✓

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13. Find the Fourier coefficient a_n for the following functions. What is the value of the Fourier series at $\theta = \pi$?

a. e^θ.

Ans. $a_n = \dfrac{(-1)^n}{2\pi}\dfrac{e^\pi - e^{-\pi}}{1 - in}$. The value of the series at $\theta = \pi$ is $\dfrac{1}{2}(e^\pi + e^{-\pi})$.

b. $\dfrac{1}{a + b\cos\theta}$.

Ans. $a_n = (-1)^n \left(a - \sqrt{a^2 - b^2}\right)^n / \left(b^n \sqrt{a^2 - b^2}\right)$ $(n > 0)$, $a_{-n} = a_n$. The value of the Fourier series at $\theta = \pi$ is $(a - b)^{-1}$.

10 points a) a_n +3

$f(\pi)$ +2

b) a_n +3

$f(\pi)$ +2

2. a).

$$f(\theta) = e^{\theta} \qquad -\pi < t \leq \pi$$

$$f(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$$

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\theta} e^{-in\theta} d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\theta(1-in)} d\theta$$

$$= \frac{1}{2\pi(1-in)}[e^{\theta}e^{-in\theta}]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi(1-in)}(e^{\pi}e^{-in\pi} - e^{-\pi}e^{in\pi})$$

$$e^{in\pi} = \cos(n\pi) + i\sin(n\pi)$$

$$\sin(n\pi) = 0 \qquad \text{for integer values of } n$$

$$\cos(n\pi) = (-1)^n \qquad \text{for integer values of } n$$

$$e^{in\pi} = (-1)^n$$

$$a_n = \frac{1}{2\pi(1-in)}(e^{\pi}(-1)^n - e^{-\pi}(-1)^n)$$

$$= \frac{(-1)^n}{2\pi}\frac{e^{\pi} - e^{-\pi}}{1-in}$$

✓ +3

$f(\pi) =$ Point of discontinuity.

At points of discontinuity, the Fourier series is the average of the value of the function from the left and right.

$$f(\pi) = \frac{e^{-\pi} + e^{\pi}}{2}$$

✓ +2

typo: $f(\pi)$ at the end should denote the fourier series at π.

b).

$$f(\theta) = \frac{1}{a + b\cos\theta} \qquad -\pi < \theta < \pi \qquad a > b > 0$$

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}d\theta$$

let $z = e^{i\theta}, C =$ counter clockwise unit circle

$$d\theta = \frac{d\theta}{iz}$$

$$\cos\theta = \frac{z^2+1}{2z}$$

$$a_n = \frac{1}{2\pi}\oint_C \frac{z^{-n}}{a + b(\frac{z^2+1}{2z})}\frac{d\theta}{iz}$$

$$= \frac{1}{\pi i}\oint_C \frac{z^{-n}}{2az + b(z^2+1)}d\theta$$

$$= \frac{1}{\pi i}\oint_C \frac{z^{-n}}{bz^2 + 2az + b}d\theta$$

For $z \leq 0,$ there are 2 simple poles:

$$z_0 = \frac{-a + \sqrt{a^2 - b^2}}{b}$$

$$z_1 = \frac{-a - \sqrt{a^2 - b^2}}{b}$$

Only z_0 is in C

$$\mathrm{Res}_{z_0}(g(z)) = \lim_{z \to z_0}(z - z_0)g(z) = \frac{{z_0^{-}}^n}{b(z_0 - z_1)}$$

$$a_{-n} = \frac{2\pi i}{\pi i}\frac{z_0^n}{b(z_0 - z_1)}$$

$$= 2(\frac{-a + \sqrt{a^2-b^2}}{b})^n \frac{b}{2b\sqrt{a^2-b^2}}$$

$$= \frac{(-1)^n(a - \sqrt{a^2-b^2})^n}{b^n\sqrt{a^2-b^2}}$$

$$\text{So, } a_n = \frac{(-1)^{|n|}(a - \sqrt{a^2-b^2})^{|n|}}{b^{|n|}\sqrt{a^2-b^2}}$$ ✓ +5

Fourier series at $\theta = \pi$:

$$F(\pi) = \frac{f(\pi) + f(-\pi)}{2} = \frac{2(a-b)^{-1}}{2} = \frac{1}{a-b}$$ ✓ +2

Problem 3

3)

a) $i\frac{\partial\psi}{\partial t} = -\frac{\partial^2\psi}{\partial x^2}$

+10/10

$i\frac{\partial\tilde{\psi}}{\partial t} = k^2\psi(k,t)$

$\psi(k,t) = a(k)e^{-ik^2t}$

$\psi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx-ik^2t} a(k)\, dk$

b) $a(k) = \tilde{f}(k)$

$\psi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-ik^2t}\tilde{f}(k)\, dk$

c) $G(x,x';t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(kx'-k^2t)} e^{-ikx} dk$

$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i(x-x'+k^2t)} dk$

$= e^{i(x-x')^2/4t}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-it(k+(x-x')/2t)^2} dk$

$= e^{i(x-x')^2/4t}\frac{1}{2\pi\sqrt{t}}\int_{-\infty}^{\infty} e^{-ik^2} dk$

$= e^{i(x-x')^2/4t}\frac{1}{2\pi\sqrt{t}}\sqrt{\pi}\, e^{-i\pi/4}$

$= \frac{e^{i(x-x')^2/4t}}{\sqrt{4\pi i t}}$

$\Rightarrow \psi(x,t) = \int_{-\infty}^{\infty} f(x')\frac{e^{i(x-x')^2/4t}}{\quad} dx'$