18.09

Pset 4

Solutions

Problem 1

$f(x)=e^{-x^2}$

$\tilde{f}(k)=\int_{-\infty}^{\infty} e^{-ikx}e^{-x^2}dx$

$=\int_{-\infty}^{\infty} e^{-(x^2+ikx)}dx = \int_{-\infty}^{\infty} e^{-(x^2+ikx+(\frac{ik}{2})^2-(\frac{ik}{2})^2)}dx = \int_{-\infty}^{\infty} e^{-((x+\frac{ik}{2})^2+\frac{k^2}{4})}dx$

$=e^{-\frac{k^2}{4}}\int_{-\infty}^{\infty} e^{-(x+\frac{ik}{2})^2}dx$, let $u=x+\frac{ik}{2} \Rightarrow du=dx$

$= e^{-\frac{k^2}{4}}\underbrace{\int_{-\infty}^{\infty} e^{-u^2}du}_{\text{gaussian integral, } =\sqrt{\pi}}$ $\Rightarrow \therefore \boxed{\tilde{f}(k)=\sqrt{\pi}e^{-\frac{k^2}{4}}}$ +2.5

inverse:

$f(x)=\int_{-\infty}^{\infty}\tilde{f}(k)\frac{e^{ikx}}{2\pi}dk$

$=\int_{-\infty}^{\infty}\frac{\sqrt{\pi}}{2\pi}e^{ikx-\frac{k^2}{4}}dk = \frac{\sqrt{\pi}}{2\pi}\int_{-\infty}^{\infty}e^{-(-ikx+\frac{k^2}{4})}dk = \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-\frac{1}{4}(k^2-4ikx)}dk$

$=\frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-\frac{1}{4}(k^2-4ixk+4i^2x^2-4i^2x^2)}dk = \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-\frac{1}{4}(k-2ix)^2-\frac{1}{4}(-4i^2x^2)}dk$

$=\frac{e^{-x^2}}{2\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-\frac{1}{4}(k-2ix)^2}dk = \frac{e^{-x^2}}{2\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-(\frac{k^2}{4}-ixk+i^2x^2)}dk$

$=\frac{e^{-x^2}}{2\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-(\frac{k}{2}-ix)^2}dk$, let $u=\frac{k}{2}-ix$

$du=\frac{1}{2}dk$

$\Rightarrow 2du=dk$

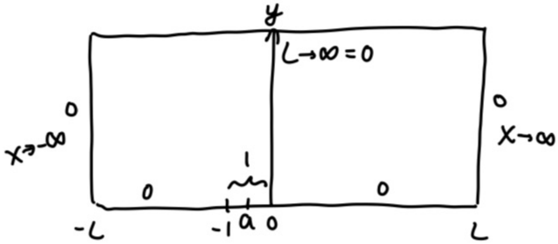
+2.5

$=\frac{e^{-x^2}}{2\sqrt{\pi}}\int_{-\infty}^{\infty}2e^{-u^2}du = \frac{e^{-x^2}}{\sqrt{\pi}}\underbrace{\int_{-\infty}^{\infty}e^{-u^2}du}_{\text{gaussian integral } =\sqrt{\pi}}$ $\Rightarrow \boxed{f(x)=e^{-x^2}}$ ✓

#2 $u = u(x,y)$ $\quad u_{xx} + u_{yy} = 0$ Satisfies Laplace Equation $y > 0$, $-\infty < x < \infty$

Boundary Conditions:

$$u(x,0) = 0 \;,\; -0 < x < \infty$$
$$= 1 \;,\; -1 < x < 0$$
$$= 0 \;,\; -\infty < x < -1$$



$u(x,y)$ w/ y constant

$$u(x,y) = \int_{-\infty}^{\infty} e^{ikx} U(k,y) \frac{dk}{2\pi} \;,\; U(k,y) = \int_{-\infty}^{\infty} e^{-ikx} u(x,y)\, dx \rightarrow (ik)^2$$

$$\nabla^2 u = 0 \quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \int_{-\infty}^{\infty} e^{ikx} U(k,y) \frac{dk}{2\pi} = \int_{-\infty}^{\infty} e^{ikx} \left[\left(-k^2 + \frac{\partial^2}{\partial y^2}\right) U(k,y)\right] = 0$$

$$= \left(-k^2 + \frac{\partial^2}{\partial y^2}\right) U(k,y) = 0 \quad U(k,y) = A(k) e^{ky} + B(k) e^{-ky} = c(k) e^{-|k|y}$$

$$u(x,y) = \int_{-\infty}^{\infty} e^{ikx} c(k) e^{-|k|y} \frac{dk}{2\pi}$$

Boundary Conditions $y = 0$ $\quad u(x,0) = f(x) = \int_{-\infty}^{\infty} e^{ikx} c(x) \frac{dk}{2\pi}$

Using Fourier Transform, $c(x) = \tilde{f}(k)$

$$u(x,y) = \int_{-\infty}^{\infty} e^{ikx} e^{-|k|y} \tilde{f}(k) \frac{dk}{2\pi} \;,\; \hat{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx = \int_{-1}^{0} e^{-ikx} dx$$

$$= \frac{-1}{ik}\left[e^{-ikx}\right]_{k=-1}^{k=0} = \frac{-1}{ik}\left[1 - e^{ix}\right] = \frac{e^{ik} - 1}{ik}$$

$$\boxed{u(x,y) = \int_{-\infty}^{\infty} e^{ikx} e^{-|k|y} \left(\frac{e^{ik}-1}{ik}\right) \frac{dk}{2\pi}}$$

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Note there were many solutions to this part