18.04

Pset 4

Answers

1

$$f(z) = \frac{1}{z(z-1)}$$

Laurent series expansion convergent for $0 < |z| < 1$

$$f(z) = -\frac{1}{z}\frac{1}{1-z} = -\frac{1}{z}(1 + z + z^2 + ...) = -\frac{1}{z} - 1 - z - z^2 - ...$$

Laurent series expansion convergent for $1 < |z|$

$$f(z) = \frac{1}{z^2(1-\frac{1}{z})} = \frac{1}{z^2}(1 + \frac{1}{z} + \frac{1}{z^2} + ...) = \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + ...$$

Pset 3
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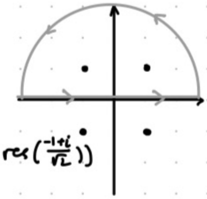
2)

a) $\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx$ singularities at $\frac{1+i}{\sqrt{2}}, \frac{-1-i}{\sqrt{2}}, \frac{1-i}{\sqrt{2}}, \frac{-1+i}{\sqrt{2}}$

equal to 0: $dz = Rie^{i\theta}$

$z^4+1 = R^4e^{4i\theta}+1$



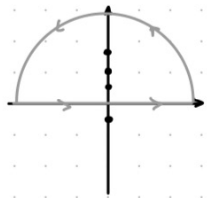
$z = Re^{i\theta}$ $\quad dz = Rie^{i\theta}$

$$\int_{-\infty}^{\infty} \frac{1}{z^4+1}\,dz = \int_{\rightarrow} \frac{1}{z^4+1}\,dz + \int_{\curvearrowleft} \frac{1}{z^4+1}\,dz = \oint \frac{1}{z^4+1}\,dz = 2\pi i \left(\mathrm{res}\left(\frac{1+i}{\sqrt{2}}\right) + \mathrm{res}\left(\frac{-1+i}{\sqrt{2}}\right)\right)$$

$$\int_{-\infty}^{\infty} \frac{1}{z^4+1}\,dz = 2\pi i\left(\frac{1+i}{4\sqrt{2}} - \frac{1-i}{4\sqrt{2}}\right) = \boxed{\frac{\pi}{\sqrt{2}}}$$

b) $\int_{-\infty}^{\infty} \frac{1}{(x-i)(x-2i)(x-3i)(x+i)}\,dx$ singularities at $i, 2i, 3i, -i$

$$X = \frac{1}{(z-i)(z-2i)(z-3i)(z+i)}$$



$z = Re^{i\theta}$ $\quad dz = Rie^{i\theta}$

$$\int_{-\infty}^{\infty} X\,dz = \int_{\rightarrow} X\,dz + \int_{\curvearrowleft} X\,dz = \oint X\,dz = 2\pi i\,(\mathrm{Res}(i) + \mathrm{Res}(2i) + \mathrm{Res}(3i))$$

$\lim_{z \to z_0} (z-z_0)X \rightarrow \mathrm{Res}(i) = \frac{i}{4}, \quad \mathrm{Res}(2i) = \frac{-i}{3}, \quad \mathrm{Res}(3i) = \frac{i}{8}$

$$\int_{-\infty}^{\infty} X\,dz = 2\pi i\left(\frac{i}{24}\right) = \boxed{\frac{-\pi}{12}}$$

c) $\int_0^{2\pi} \frac{1}{5+\cos\theta}\,d\theta$

$e^{i\theta} = z \rightarrow ie^{i\theta}\,d\theta = dz$

$iz\,d\theta = dz$

$d\theta = \frac{1}{zi}\,dz$

$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z^2+1}{2z}$

$$= \int_0^{2\pi} \frac{1}{5+\frac{z^2+1}{2z}} \cdot \frac{1}{zi}\,dz$$

$$= \int_0^{2\pi} \frac{2z}{10z+z^2+1} \cdot \frac{1}{iz}\,dz$$

$$= \int_0^{2\pi} \frac{-2i}{10z+z^2+1}\,dz = (-2i)\int_0^{2\pi} \frac{1}{10z+z^2+1}\,dz$$

singularities: $-5 \pm 2\sqrt{6}$

since $-5-2\sqrt{6} < -2\pi$, ignore

$$= (-2i)\int_0^{2\pi} \frac{1}{10z+z^2+1}\,dz = -2i\left(2\pi i\left[\mathrm{Res}(-5+2\sqrt{6})\right]\right)$$

$\mathrm{Res}(-5+2\sqrt{6}) = \frac{1}{4\sqrt{6}}$

$$= -2i \cdot 2\pi i \cdot \frac{1}{4\sqrt{6}} = \boxed{\frac{\pi}{\sqrt{6}}}$$

3a) Note people had very different answers since the original problem wasn't solvable so I didn't grade.

#3b $I=\int_0^\infty \frac{x\sin x}{1+x^2}\,dx$ singularity at $z=i$ $\quad I=\frac{1}{2}\operatorname{Im}\int_{-\infty}^{\infty}\frac{ze^{iz}}{1+z^2}\,dz = \operatorname{Im} J$

even function

$$= \operatorname{Im} 2\pi i \lim_{z\to i}\frac{ze^{iz}}{(z+i)} = \operatorname{Im} 2\pi i\,\frac{ie^{i^2}}{2i} = \operatorname{Im}\frac{\pi i}{2e} = \boxed{\frac{\pi}{2e}}$$

#3c $\int_{-\infty}^{\infty}\frac{\sin^2 x}{x^2}\,dx = \int_{-\infty}^{\infty}\left(\frac{\frac{e^{iz}-e^{-iz}}{2i}}{z}\right)^2 dz = \int \frac{1}{-4z^2}\left(\cancel{e^{2iz}}^{\,0}+e^{-2iz}-2\right)dz$

$$= 2\pi i\left(-\lim_{z\to 0}\frac{1}{(2-1)!}\frac{d}{dz}(z-0)^2\,\frac{e^{-2iz}-2}{-4z^2}\right) = 2\pi i\lim_{z\to 0}\frac{d}{dz}\frac{e^{-2iz}-2}{4}$$

$$= \frac{1}{2}\pi i\lim_{z\to 0}\left(-2ie^{-2iz}\right) = \frac{1}{2}\pi i\,(-2i) = \boxed{\pi}$$