18.09 Pset 2

Solutions

PSET #2 Khang Le

1) i^{2i}

$$i = e^{i(2\pi k + \frac{\pi}{2})} \qquad k \in \mathbb{Z}$$

$$\Rightarrow i^{2i} = e^{i(2\pi k + \frac{\pi}{2}) \cdot 2i}$$

$$= e^{-2(2\pi k + \frac{\pi}{2})}$$

$$= e^{-(4\pi k + \pi)} \qquad k \in \mathbb{Z}$$

$$\Rightarrow i^{2i} = \begin{cases} e^{7\pi} & k = -2 \\ e^{3\pi} & k = -1 \\ e^{-\pi} & k = 0 \\ e^{-5\pi} & k = 1 \\ e^{-(4\pi k + \pi)} & k = \mathbb{Z} \end{cases}$$

2) $(z+1)^4 = (z^2-1)^4$

$$\Rightarrow (z+1)e^{i\frac{2\pi n}{4}} = (z^2-1)$$

$$\Rightarrow (z+1)(z-1-e^{i2\pi n/4}) = 0$$

$$ze^{i\frac{2\pi n}{4}} + e^{i\frac{2\pi n}{4}} = z^2 - 1$$

$$-z^2 + ze^{i\frac{2\pi n}{4}} + e^{i\frac{2\pi n}{4}} + 1 = 0$$

$$\Rightarrow z^2 - ze^{i2\pi n/4} - e^{i2\pi n/4} - 1 = 0$$

$$\Rightarrow (z+1)(z-1-e^{i2\pi n/4}) = 0 \qquad n = 0, 1, 2, 3$$

$$\Rightarrow z = -1$$

$$z = 1 + e^{i2\pi n/4} \qquad \Rightarrow n = 0, 1, 2, 3$$

$$\Rightarrow z = -1, 0, 2, 1-i, 1+i$$

Problem 2:

$u = xy^2 + ax^3$ $\quad u_x = y^2 + 3ax^2 = v_y \Rightarrow v = \frac{y^3}{3} + 3ax^2y + g(x) \Rightarrow$ $\quad 6axy = -2xy$ only if

$f(z) = u + iv$ $\quad -u_y = -2xy = v_x \quad v_x = 6axy + g'(x) = -2xy$

Analytic if

$u_x = v_y$ $\quad -2xy + g'(x) = -2xy$ $\quad \boxed{a = \frac{-1}{3}}$

$u_y = -v_x$ $\quad g'(x) = 0 \Rightarrow g(x) = c$

+10/10

$$u = xy^2 - \frac{x^3}{3}, \quad \boxed{v = \frac{y^3}{3} - x^2y + c}$$

$$f(z) = u + iv = \left(xy^2 - \frac{x^3}{3}\right) + i\left(\frac{y^3}{3} - x^2y + c\right) \qquad z = x + iy$$

$$f(z) = \frac{-1}{3}(x+iy)^3 = \frac{-1}{3}(x^3 + 3x^2yi - 3xy^2 - iy^3)$$

$$= \left(\frac{-1}{3}x^3 + xy^2\right) + i\left(\frac{y^3}{3} - x^2y\right)$$

$$\boxed{f(z) = \frac{-1}{3}z^3}$$
