18.09 Pset 1

Solutions

[18.04] pset 1

1.1

$$\textbf{solving for } Y$$

$$\text{let } y_p \text{ satisfy: } (\frac{d^4}{dx^4} - 1)y_p = \sin 2x + 2x + e^{3x}$$

$$(\frac{d^4}{dx^4} - 1)y = (\frac{d^4}{dx^4} - 1)(y_p + Y) = \sin 2x + 2x + e^{3x} \implies (\frac{d^4}{dx^4} - 1)Y = 0$$

$$\text{try } Y = e^{mx}$$

$$(D^4 - 1)Y = (D^4 - 1)e^{mx} = m^4 e^{mx} - e^{mx} \implies m = \pm 1, \pm i$$

$$Y = c_1 e^x + c_2 e^{-x} + c_3 e^{ix} + c_4 e^{-ix}$$

$$\textbf{solving for } y_p$$

$$y_p = y_{p1} + y_{p2} + y_{p3} \text{ where } \frac{d^4 y_{p1}}{dx^4} - y_{p1} = \sin 2x \text{ and } \frac{d^4 y_{p2}}{dx^4} - y_{p2} = 2x \text{ and } \frac{d^4 y_{p3}}{dx^4} - y_{p3} = e^{3x}$$

$$---$$

$$\frac{d^4 y_{p1}}{dx^4} - y_{p1} = \sin 2x$$

$$\text{try } y_{p1} = a \sin(2x) + b \cos(2x)$$

$$\frac{d^4 y_{p1}}{dx^4} = 16a \sin(2x) + 16b \cos(2x)$$

$$\frac{d^4 y_{p1}}{dx^4} - y_{p1} = \sin 2x = 16a \sin(2x) + 16b \cos(2x) - (a \sin(2x) + b \cos(2x))$$

$$b \text{ must be } 0, \text{ solving for a yields } a = 15$$

$$y_{p1} = \frac{\sin 2x}{15}$$

$$---$$

$$\frac{d^4 y_{p2}}{dx^4} - y_{p2} = 2x \implies y_{p2} = -2x$$

$$---$$

$$\frac{d^4 y_{p3}}{dx^4} - y_{p3} = e^{3x}$$

$$y_{p3} = \frac{1}{D^4 - 1} e^{3x} = \frac{1}{3^4 - 1} e^{3x} = \frac{1}{80} e^{3x}$$

$$---$$

$$y_p = \frac{\sin 2x}{15} - 2x + \frac{e^{3x}}{80}$$

$$\textbf{putting it together}$$

$$y = Y + y_p = c_1 e^x + c_2 e^{-x} + c_3 e^{ix} + c_4 e^{-ix} + \frac{\sin 2x}{15} - 2x + \frac{e^{3x}}{80}$$ +5/5

1.2

$$\frac{d^2y}{dx^2} - y = 2 + x^2 + e^x$$

$$\text{general solution: } y = y_p + Y$$

solving for Y

$$\text{let } y_p \text{ satisfy: } (\frac{d^2}{dx^2} - 1)y_p = 2 + x^2 + e^x$$

$$(\frac{d^2}{dx^2} - 1)y = (\frac{d^2}{dx^2} - 1)(y_p + Y) = 2 + x^2 + e^x \implies (\frac{d^2}{dx^2} - 1)Y = 0$$

$$try \quad Y = e^{mx}$$

$$(D^2 - 1)Y = (D^2 - 1)e^{mx} = m^2e^{mx} - e^{mx} \implies m = \pm 1$$

$$Y = c_1e^x + c_2e^{-x}$$

solving for y_p

$$y_p = y_{p1} + y_{p2} + y_{p3} \text{ where } \frac{d^2y_{p1}}{dx^2} - y_{p1} = 2 \text{ and } \frac{d^2y_{p2}}{dx^2} - y_{p2} = x^2 \text{ and } \frac{d^2y_{p3}}{dx^2} - y_{p3} = e^x$$

$$\frac{d^2y_{p1}}{dx^2} - y_{p1} = 2 \implies y_{p1} = -2$$

$$\frac{d^2y_{p2}}{dx^2} - y_{p2} = x^2 \implies y_{p2} = \frac{1}{D^2-1}x^2 = -\frac{1}{1-D^2}x^2 = -(1 + D^2 + D^4 + \dots)x^2 = -(x^2+2)$$

$$\frac{d^2y_{p3}}{dx^2} - y_{p3} = e^x \implies y_{p3} = \frac{1}{D^2-1}e^x$$

$$\frac{1}{D^2-1} = \frac{1}{-2}\frac{1}{D+1} + \frac{1}{2}\frac{1}{D-1}$$

$$y_{p3} = -\frac{1}{2}\frac{1}{D+1}e^x + \frac{1}{2}\frac{1}{D-1}e^x$$

$$\text{using } \frac{d}{dt}(e^{mt}u) = e^{mt}(\frac{d}{dt} + m)u$$

$$y_{p3} = -\frac{1}{4}e^x + \frac{1}{2}e^x\frac{1}{D-1+1} = -\frac{1}{4}e^x + \frac{1}{2}e^x x$$

$$y_p = -2 + -(x^2+2) + -\frac{1}{4}e^x + \frac{1}{2}e^x x = -4 - x^2 - \frac{1}{4}e^x + \frac{1}{2}e^x x$$

putting it together

$$y = y_p + Y = -4 - x^2 - \frac{1}{4}e^x + \frac{1}{2}e^x x + c_1e^x + c_2e^{-x}$$

$$= -x^2 + \frac{e^x x}{2} - 4 + c_1e^x + c_2e^{-x}$$

+5/5

Problem 2

$$\frac{d^2y}{dx^2} - y = 0 \rightarrow (D^2-1)y = 0 \rightarrow (m^2-1)=0 \rightarrow m = \pm 1$$

$$y = ae^{x} + be^{-x}$$

$$y(0) = 0 = a+b \qquad \rightarrow \qquad a = -b$$
$$y'(0) = 1 = a-b \qquad\qquad 1 = -2b \rightarrow \boxed{b = -\tfrac{1}{2} \rightarrow a = \tfrac{1}{2}}$$

$$\boxed{y = \tfrac{1}{2}e^{x} - \tfrac{1}{2}e^{-x}}$$

+10

Problem 3

1.a. $x(1+x)y' + \frac{1+x}{2}y = \sqrt{x}$

$$x(1+x)\frac{dy}{dx} + \frac{1+x}{2}y = x^{1/2}$$

$$\frac{dy}{dx} + \frac{1}{2x}y = \frac{1}{x^{1/2}(1+x)}$$

$$\rightarrow u(x) = \frac{1}{2x}, \quad w(x) = \frac{1}{x^{1/2}+x^{3/2}}$$

$$y(x) = e^{-\int u dx}\left(A + \int w e^{\int u dx}\right)$$

$$\int u dx = \int \frac{1}{2x}dx = \frac{1}{2}\int \frac{1}{x}dx = \frac{1}{2}\ln|x| + c$$

$$y(x) = e^{-(\frac{1}{2}\ln|x|+c)}\left(A + \int w e^{(\frac{1}{2}\ln|x|+c)}\right)$$

$$= Ae^{-\frac{1}{2}\ln|x|}e^{-c} + e^{-\frac{1}{2}\ln|x|}e^{-c}\left(e^{c}\int w e^{\frac{1}{2}\ln|x|}dx\right)$$

$$= Ae^{-\frac{1}{2}\ln|x|}e^{-c} + e^{-\frac{1}{2}\ln|x|}\int \frac{e^{\frac{1}{2}\ln|x|}}{x^{1/2}+x^{3/2}}\,dx$$

$$= Ae^{-\frac{1}{2}\ln|x|}e^{-c} + e^{-\frac{1}{2}\ln|x|}\int \frac{x^{1/2}}{x^{1/2}+x^{3/2}}\,dx$$

$$= Ae^{-\frac{1}{2}\ln|x|}e^{-c} + e^{-\frac{1}{2}\ln|x|}\int \frac{1}{1+x}\,dx$$

let $u = 1+x \to du = dx$

$$= Ax^{-\frac{1}{2}}e^{-c} + x^{-\frac{1}{2}}\int \frac{1}{u}\,du = Ax^{-\frac{1}{2}}e^{-c} + x^{-\frac{1}{2}}(\ln|u|+k) = Ax^{-\frac{1}{2}}e^{-c} + x^{-\frac{1}{2}}(\ln|1+x|+k)$$

$$= \left(Ae^{-c} + \ln(1+x) + k\right)x^{-\frac{1}{2}}$$

$$y = \left(A + \ln(1+x)\right)x^{-\frac{1}{2}} \qquad y' = -\frac{A}{2}x^{-\frac{3}{2}} + \frac{x^{-1/2}}{1+x} - \frac{1}{2}x^{-3/2}\ln(1+x)$$

$$x(1+x)y' + \frac{1+x}{2}y = \sqrt{x}$$

$$\cancel{-\frac{A}{2}(1+x)x^{-\frac{1}{2}}} + x^{\frac{1}{2}} \cancel{- \frac{1}{2}x^{-\frac{1}{2}}(1+x)\ln(1+x)} \cancel{+ \frac{A}{2}(1+x)x^{-\frac{1}{2}}} \cancel{+ \frac{1}{2}(1+x)\ln(1+x)x^{-1/2}} = x^{1/2}$$

$$\to \boxed{y = \left(A + \ln(1+x)\right)x^{-\frac{1}{2}}}$$