7. Beats

7.1. **What beats are.** Beats occur when two very nearby pitches are sounded simultaneously. Musicians tune their instruments using beats. They are also used in reconstructing an amplitude-modulated signal from a frequency-modulated ("FM") radio signal. The radio receiver produces a signal at a fixed frequency ν, and adds it to the received signal, whose frequency differs slightly from ν. The result is a beat, and the beat frequency is the audible frequency.

We'll make a mathematical study of this effect, using complex numbers.

We will study the sum of two sinusoidal functions. We might as well take one of them to be $a\sin(\omega_0 t)$, and adjust the phase of the other accordingly. So the other can be written as $b\sin((1+\epsilon)\omega_0 t - \phi)$: amplitude b, angular frequency written in terms of the frequency of the first sinusoid as $(1+\epsilon)\omega_0$, and phase lag ϕ.

We will take $\phi = 0$ for the moment, and add it back in later. So we are studying

$$x = a\sin(\omega_0 t) + b\sin((1+\epsilon)\omega_0 t).$$

We think of ϵ as a small number, so the two frequencies are relatively close to each other.

One case admits a simple discussion, namely when the two amplitudes are equal: $a = b$. Then the trig identity

$$\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\cos(\beta)\sin(\alpha)$$

with $\alpha = (1+\epsilon/2)\omega_0 t$ and $\beta = \epsilon\omega_0 t/2$ gives us the equation

$$x = a\sin(\omega_0 t) + a\sin((1+\epsilon)\omega_0 t) = 2a\cos\left(\frac{\epsilon\omega_0 t}{2}\right)\sin\left(\left(1+\frac{\epsilon}{2}\right)\omega_0 t\right).$$

(The trig identity is easy to prove using complex numbers: Compute

$$e^{i(\alpha+\beta)} + e^{i(\alpha-\beta)} = (e^{i\beta} + e^{-i\beta})e^{i\alpha} = 2\cos(\beta)e^{i\alpha}$$

using (6.5); then take imaginary parts.)

We might as well take $a > 0$. When ϵ is small, the period of the cosine factor is much longer than the period of the sine factor. This lets us think of the product as a wave of angular frequency $(1+\epsilon/2)\omega_0$—that is, the average of the angular frequences of the two constituent waves—giving the audible tone, whose amplitude is modulated by multiplying it by

$$g(t) = 2a \left| \cos\left(\frac{\epsilon\omega_0 t}{2}\right) \right| . \tag{1}$$

The function $g(t)$ the "envelope" of x. The function $x(t)$ oscillates rapidly between $-g(t)$ and $+g(t)$.

To study the more general case, in which a and b differ, we will study the function made of complex exponentials,

$$z = ae^{i\omega_0 t} + be^{i(1+\epsilon)\omega_0 t}.$$

The original function x is the imaginary part of z.

We can factor out $e^{i\omega_0 t}$:

$$z = e^{i\omega_0 t}(a + be^{i\epsilon\omega_0 t}).$$

This gives us a handle on the magnitude of z, since the magnitude of the first factor is 1. Using the formula $|w|^2 = w\bar{w}$ on the second factor, we get

$$|z|^2 = a^2 + b^2 + 2ab\cos(\epsilon\omega_0 t).$$

The imaginary part of a complex number z lies between $-|z|$ and $+|z|$, so $x = \operatorname{Im} z$ oscillates between $-|z|$ and $+|z|$. The function $g(t) = |z(t)|$, i.e.

$$g(t) = \sqrt{a^2 + b^2 + 2ab\cos(\epsilon\omega_0 t)}, \tag{2}$$

thus serves as an "envelope," giving the values of the peaks of the oscillations exhibited by $x(t)$.

This envelope shows the "beats" effect. It reaches maxima when $\cos(\epsilon\omega_0 t)$ does, i.e. at the times $t = 2k\pi/\epsilon\omega_0$ for whole numbers k. A single beat lasts from one maximum to the next: the period of the beat is

$$P_b = \frac{2\pi}{\epsilon\omega_0} = \frac{P_0}{\epsilon}$$

where $P_0 = 2\pi/\omega_0$ is the period of $\sin(\omega_0 t)$. The maximum amplitude is then $a + b$, i.e. the sum of the amplitudes of the two constituent waves; this occurs when their phases are lined up so they reinforce. The minimum amplitude occurs when the cosine takes on the value -1, i.e. when $t = (2k + 1)\pi/\epsilon\omega_0$ for whole numbers k, and is $|a - b|$. This is when the two waves are perfectly out of sync, and experience destructive interference.

Figure 4 is a plot of beats with $a = 1, b = .5, \omega_0 = 1, \epsilon = .1, \phi = 0$, showing also the envelope.

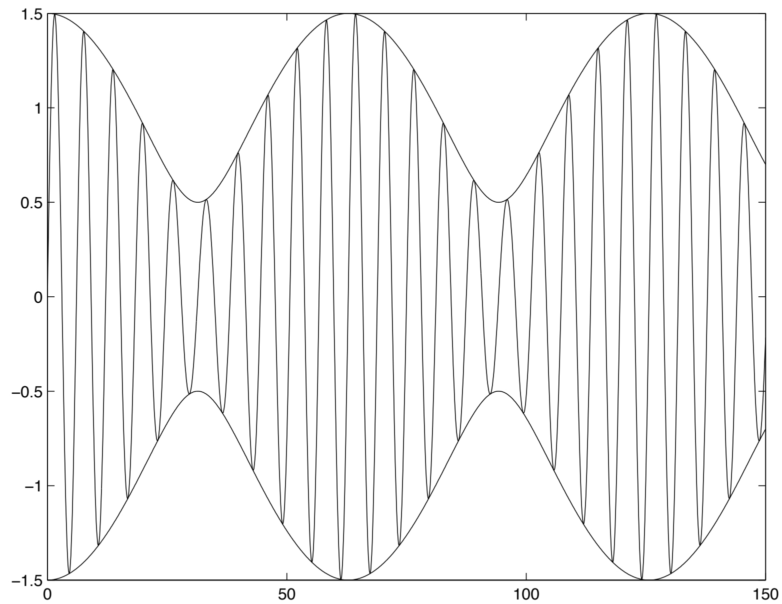

FIGURE 4. Beats, with envelope

Now let's allow ϕ to be nonzero. The effect on the work done above is to replace $\epsilon\omega_0 t$ by $\epsilon\omega_0 t - \phi$ in the formulas (2) for the envelope $g(t)$. Thus the beat gets shifted by the same phase as the second signal.

If $b \neq 1$ it is not very meaningful to compute the pitch, i.e. the frequency of the wave being modulated by the envelope. It lies somewhere between the two initial frequencies, and it varies periodically with period P_b.

7.2. **What beats are not.** Many differential equations textbooks present beats as a system response when a harmonic oscillator is driven by a signal whose frequency is close to the natural frequency of the oscillator. This is true as a piece of mathematics, but it is almost never the way beats occur in nature. The reason is that if there is any damping in the system, the "beats" die out very quickly to a steady sinusoidal solution, and it is that solution which is observed.

Explicitly, the Exponential Response Formula (Section 14, equation 3) shows that the equation

$$\ddot{x} + \omega_n^2 x = \cos(\omega t)$$

has the periodic solution

$$x_p = \frac{\cos(\omega t)}{\omega^2 - \omega_n^2}$$

unless $\omega = \omega_n$. If ω and ω_n are close, the amplitude of the periodic solution is large; this is "near resonance." Adding a little damping won't change that solution very much, but it will convert homogeneous solutions from sinusoids to *damped* sinusoids, i.e. transients, and rather quickly any solution becomes indistinguishable from x_p. So beats do not occur this way in engineering situations.

Differential equations textbooks also always arrange initial conditions in a very artificial way, so that the solution is a sum of the periodic solution x_p and a homogeneous solution x_h having exactly the same amplitude as x_p. They do this by imposing the initial condition $x(0) = \dot{x}(0) = 0$. This artifice puts them into the simple situation $a = b$ mentioned above. For the general case one has to proceed as we did, using complex exponentials.