6. The complex exponential

The exponential function is a basic building block for solutions of ODEs. Complex numbers expand the scope of the exponential function, and bring trigonometric functions under its sway.

6.1. **Exponential solutions.**

The function e^t is *defined* to be the solution of the initial value problem $\dot{x} = x$, $x(0) = 1$. More generally, the chain rule implies the

Exponential Principle:

For any constant w, e^{wt} is the solution of $\dot{x} = wx$, $x(0) = 1$.

Now look at a more general constant coefficient homogeneous linear ODE, such as the second order equation

$$\ddot{x} + c\dot{x} + kx = 0. \tag{1}$$

It turns out that there is always a solution of (1) of the form $x = e^{rt}$, for an appropriate constant r.

To see what r should be, take $x = e^{rt}$ for an as yet to be determined constant r, substitute it into (1), and apply the Exponential Principle. We find

$$(r^2 + cr + k)e^{rt} = 0.$$

Cancel the exponential (which, conveniently, can never be zero), and discover that r must be a root of the polynomial $p(s) = s^2+cs+k$. This is the characteristic polynomial of the equation. The **characteristic polynomial** of the linear equation with constant coefficients

$$a_n\frac{d^nx}{dt^n} + \cdots + a_1\frac{dx}{dt} + a_0x = 0$$

is

$$p(s) = a_ns^n + \cdots + a_1s + a_0\,.$$

Its roots are the **characteristic roots** of the equation. We have discovered the

Characteristic Roots Principle:

(2)	e^{rt} is a solution of a constant coefficient homogeneous linear differential equation exactly when r is a root of the characteristic polynomial.

Since most quadratic polynomials have two distinct roots, this normally gives us two linearly independent solutions, e^{r_1t} and e^{r_2t}. The general solution is then the linear combination $c_1e^{r_1t} + c_2e^{r_2t}$.

This is fine if the roots are real, but suppose we have the equation

$$\ddot{x} + 2\dot{x} + 2x = 0 \tag{3}$$

for example. By the quadratic formula, the roots of the characteristic polynomial $s^2 + 2s + 2$ are the complex conjugate pair $-1 \pm i$. We had better figure out what is meant by $e^{(-1+i)t}$, for our use of exponentials as solutions to work.

6.2. **The complex exponential.** We don't yet have a definition of e^{it}. Let's hope that we can define it so that the Exponential Principle holds. This means that it should be the solution of the initial value problem

$$\dot{z} = iz\,, \quad z(0) = 1\,.$$

We will probably have to allow it to be a *complex valued* function, in view of the i in the equation. In fact, I can produce such a function:

$$z = \cos t + i \sin t\,.$$

Check: $\dot{z} = -\sin t + i\cos t$, while $iz = i(\cos t + i\sin t) = i\cos t - \sin t$, using $i^2 = -1$; and $z(0) = 1$ since $\cos(0) = 1$ and $\sin(0) = 0$.

We have now justified the following definition, which is known as

Euler's formula:

$$\boxed{e^{it} = \cos t + i \sin t} \tag{4}$$

In this formula, the left hand side is *by definition* the solution to $\dot{z} = iz$ such that $z(0) = 1$. The right hand side writes this function in more familiar terms.

We can reverse this process as well, and express the trigonometric functions in terms of the exponential function. First replace t by $-t$ in (4) to see that

$$e^{-it} = \overline{e^{it}}\,.$$

Then put $z = e^{it}$ into the formulas (5.1) to see that

$$\boxed{\cos t = \frac{e^{it} + e^{-it}}{2}\,, \quad \sin t = \frac{e^{it} - e^{-it}}{2i}} \tag{5}$$

We can express the solution to

$$\dot{z} = (a + bi)z\,, \quad z(0) = 1$$

in familiar terms as well: I leave it to you to check that it is

$$z = e^{at}(\cos(bt) + i\sin(bt)).$$

We have discovered what e^{wt} must be, if the Exponential principle is to hold true, for any complex constant $w = a + bi$:

$$\boxed{e^{(a+bi)t} = e^{at}(\cos bt + i \sin bt)} \tag{6}$$

The complex number

$$e^{i\theta} = \cos\theta + i\sin\theta$$

is the point on the unit circle with polar angle θ.

Taking $t = 1$ in (6), we have

$$e^{a+ib} = e^a(\cos b + i \sin b)\,.$$

This is the complex number with polar coordinates e^a and b: its modulus is e^a and its argument is b. You can regard the complex exponential as nothing more than a notation for a complex number in terms of its polar coordinates. If the polar coordinates of z are r and θ, then

$$z = e^{\ln r + i\theta}$$

Exercise 6.2.1. Find expressions of $1, i, 1+i$, and $(1+\sqrt{3}i)/2$, as complex exponentials.

6.3. **Real solutions.** Let's return to the example (3). The root $r_1 = -1 + i$ leads to

$$e^{(-1+i)t} = e^{-t}(\cos t + i \sin t)$$

and $r_2 = -1 - i$ leads to

$$e^{(-1-i)t} = e^{-t}(\cos t - i \sin t)\,.$$

We probably really wanted a *real* solution to (3), however. For this we have the

Reality Principle:

(7) | If z is a solution to a homogeneous linear equation with real coefficients, then the real and imaginary parts of z are too. |

We'll explain why this is true in a minute, but first let's look at our example (3). The real part of $e^{(-1+i)t}$ is $e^{-t}\cos t$, and the imaginary part is $e^{-t}\sin t$. Both are solutions to (3), and the general real solution is a linear combination of these two.

In practice, you should just use the following consequence of what we've done:

Real solutions from complex roots:

If $r_1 = a + bi$ is a root of the characteristic polynomial of a homogeneous linear ODE whose coefficients are constant and real, then $$e^{at}\cos(bt) \quad \text{and} \quad e^{at}\sin(bt)$$ are solutions. If $b \neq 0$, they are independent solutions.

To see why the Reality Principle holds, suppose z is a solution to a homogeneous linear equation with real coefficients, say

$$\ddot{z} + p\dot{z} + qz = 0 \tag{8}$$

for example. Let's write x for the real part of z and y for the imaginary part of z, so $z = x + iy$. Since q is real,

$$\mathrm{Re}\,(qz) = qx \quad \text{and} \quad \mathrm{Im}\,(qz) = qy.$$

Derivatives are computed by differentiating real and imaginary parts separately, so (since p is also real)

$$\mathrm{Re}\,(p\dot{z}) = p\dot{x} \quad \text{and} \quad \mathrm{Im}\,(p\dot{z}) = p\dot{y}.$$

Finally,

$$\mathrm{Re}\,\ddot{z} = \ddot{x} \quad \text{and} \quad \mathrm{Im}\,\ddot{z} = \ddot{y}$$

so when we break down (8) into real and imaginary parts we get

$$\ddot{x} + p\dot{x} + qx = 0\,, \quad \ddot{y} + p\dot{y} + qy = 0$$

—that is, x and y are solutions of the same equation (8).

6.4. **Multiplication.** Multiplication of complex numbers is expressed very beautifully in these polar terms. We already know that

(9) **Magnitudes Multiply:** $|wz| = |w||z|$.

To understand what happens to arguments we have to think about the product $e^r e^s$, where r and s are two complex numbers. This is a major test of the reasonableness of our definition of the complex exponential, since we know what this product ought to be (and what it *is* for r and s real). It turns out that the notation is well chosen:

Exponential Law:

(10) | For any complex numbers r and s, $e^{r+s} = e^r e^s$ |

This fact comes out of the uniqueness of solutions of ODEs. To get an ODE, let's put t into the picture: we claim that

$$e^{r+st} = e^r e^{st}. \tag{11}$$

If we can show this, then the Exponential Law as stated is the case $t = 1$. Differentiate each side of (11), using the chain rule for the left hand side and the product rule for the right hand side:

$$\frac{d}{dt}e^{r+st} = \frac{d(r+st)}{dt}e^{r+st} = se^{r+st}\,, \quad \frac{d}{dt}(e^r e^{st}) = e^r \cdot se^{st}.$$

Both sides of (11) thus satisfy the IVP

$$\dot{z} = sz\,, \quad z(0) = e^r,$$

so they are equal.

In particular, we can let $r = i\alpha$ and $s = i\beta$:

$$e^{i\alpha}e^{i\beta} = e^{i(\alpha+\beta)}. \tag{12}$$

In terms of polar coordinates, this says that

$$\textbf{Angles Add:} \qquad \mathrm{Arg}(wz) = \mathrm{Arg}(w) + \mathrm{Arg}(z). \tag{13}$$

Exercise 6.4.1. Compute $((1+\sqrt{3}i)/2)^3$ and $(1+i)^4$ afresh using these polar considerations.

Exercise 6.4.2. Derive the addition laws for cosine and sine from Euler's formula and (12). Understand this exercise and you'll never have to remember those formulas again.

6.5. **Roots of unity and other numbers.** The polar expression of multiplication is useful in finding roots of complex numbers. Begin with the sixth roots of 1, for example. We are looking for complex numbers z such that $z^6 = 1$. Since *moduli multiply*, $|z|^6 = |z^6| = |1| = 1$, and since moduli are nonnegative this forces $|z| = 1$: all the sixth roots of 1 are on the unit circle. *Arguments add*, so the argument of a sixth root of 1 is an angle θ so that 6θ is a multiple of 2π (which are the angles giving 1). Up to addition of multiples of 2π there are six such angles: $0, \pi/3, 2\pi/3, \pi, 4\pi/3,$ and $5\pi/3$. The resulting points on the unit circle divide it into six equal arcs. From this and some geometry or trigonometry it's easy to write down the roots as $a + bi$: ± 1 and $(\pm 1 \pm \sqrt{3}i)/2$. In general, the nth roots of 1 break the circle evenly into n parts.

Exercise 6.5.1. Write down the eighth roots of 1 in the form $a + bi$.

Now let's take roots of numbers other than 1. Start by finding a single nth root z of the complex number $w = re^{i\theta}$ (where r is a positive real number). Since magnitudes multiply, $|z| = \sqrt[n]{r}$. Since angles add, one choice for the argument of z is θ/n: one nth of the way up from the positive real axis. Thus for example one square root of $4i$ is the complex number with magnitude 2 and argument $\pi/4$, which is $\sqrt{2}(1+i)$. To get all the nth roots of w notice that you can multiply one by any nth root of 1 and get another nth root of w. Angles add and magnitudes multiply, so the effect of this is just to add a multiple of $2\pi/n$ to the angle of the first root we found. There are n distinct nth roots of any nonzero complex number $|w|$, and they divide the circle with center 0 and radius $\sqrt[n]{r}$ evenly into n arcs.

Exercise 6.5.2. Find all the cube roots of -8. Find all the sixth roots of $-i/64$.

We can use our ability to find complex roots to solve more general polynomial equations.

Exercise 6.5.3. Find all the roots of the polynomials x^3+1, $ix^2+x+(1+i)$, and x^4-2x^2+1.

6.6. **Spirals.** As t varies, the complex-valued function

$$e^{it} = \cos t + i \sin t$$

parametrizes the unit circle in the complex plane. As t increases from 0 to 2π, the complex number $\cos t + i \sin t$ moves once counterclockwise around the circle.

More generally, for fixed real a, b,

$$z(t) = e^{(a+bi)t} = e^{at}(\cos(bt) + i\sin(bt)). \tag{14}$$

parametrizes a curve in the complex plane. What is it? The `Complex Exponential` Mathlet illustrates this.

When $t = 0$ we get $z(0) = 1$ no matter what a and b are.

The modulus of $z(t)$ is $|z(t)| = e^{at}$. When $a > 0$ this is increasing exponentially as t increases; when $a < 0$ it is decreasing exponentially.

Meanwhile, the other term, $\cos(bt) + i\sin(bt)$, is (for $b > 0$) winding counterclockwise around the unit circle with angular frequency b.

The product will thus parametrize a *spiral*, it runing away from the origin exponentially if $a > 0$ and decaying exponentially if $a < 0$, and winding counterclockwise if $b > 0$ and clockwise if $b < 0$. If $a = 0$ equation (14) parametrizes a circle. If $b = 0$, the curve lies on the positive real axis.

Figure 3 shows a picture of the curve parametrized by $e^{(1+2\pi i)t}$.

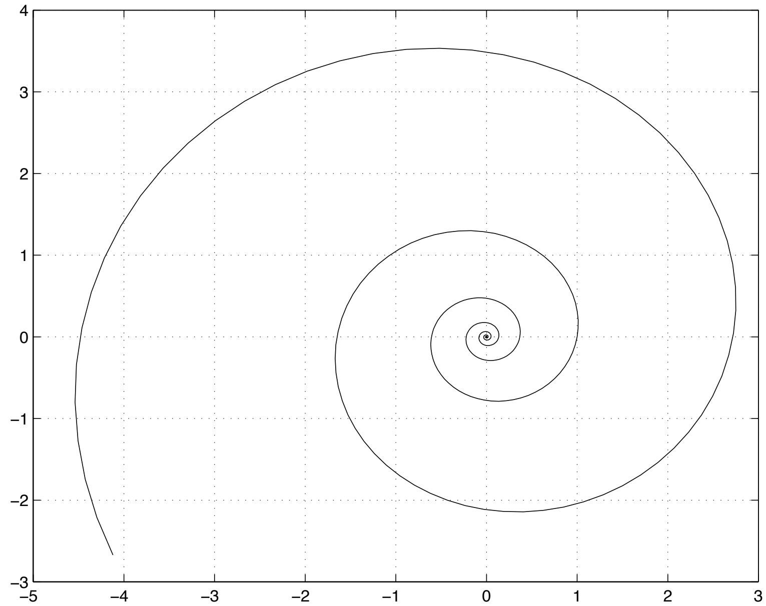

FIGURE 3. The spiral $z = e^{(1+2\pi i)t}$