31. Phase portraits in two dimensions

This section presents a very condensed summary of the behavior of two dimensional linear systems, followed by a catalogue of linear phase portraits. A much richer understanding of this gallery can be achieved using the Mathlets `Linear Phase Portraits: Cursor Entry` and `Linear Phase Portraits: Matrix Entry`.

31.1. **Phase portraits and eigenvectors.**

It is convenient to represent the solutions of an autonomous system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ (where $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$) by means of a *phase portrait.* The x, y plane is called the *phase plane* (because a point in it represents the state or phase of a system). The phase portrait is a representative sampling of trajectories of the system. A *trajectory* is the directed path traced out by a solution. It does *not* include information about the time at which solutions pass through various points (which will depend upon when the clock was set), nor does it display the speed at which the solution passes through the point—only the direction of travel. Still, it conveys essential information about the qualitative behavior of solutions of the system of equations.

The building blocks for the phase portrait of a general system will be the phase portraits of *homogeneous linear constant coefficient* systems: $\dot{\mathbf{x}} = A\mathbf{x}$, where A is a constant square matrix. Notice that this equation *is* autonomous!

The phase portraits of these linear systems display a startling variety of shapes and behavior. We'll want names for them, and the names I'll use differ slightly from the names used in the book and in some other sources.

One thing that can be read off from the phase portrait is the stability properties of the system. A linear autonomous system is *unstable* if most of its solutions tend to infinity with time. (The meaning of "most" will become clearer below.) It is *asymptotically stable* if all of its solutions tend to 0 as t goes to ∞. Finally it is *neutrally stable* if none of its solutions tend to infinity with time but most of them do not tend to zero either. It is an interesting fact that any linear autonomous system exhibits one of these three behaviors.

The *characteristic polynomial* of a square matrix A is defined to be

$$p_A(s) = \det(A - sI).$$

If A is $n \times n$, this polynomial has the following form:

$$p_A(s) = (-s)^n + (\mathrm{tr}A)(-s)^{n-1} + \cdots + (\det A),$$

where the dots represent less familiar combinations of the entries of A. When A is 2×2, say $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, this reads

$$p_A(s) = s^2 - (\mathrm{tr}A)s + (\det A).$$

We remind the reader that in this case, when

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

$$\mathrm{tr}A = a + d\,, \quad \det A = ad - bc.$$

From the eigenvalues we may reconstruct $\mathrm{tr}A$ and $\det A$, since

$$p_A(s) = (s - \lambda_1)(s - \lambda_2) = s^2 - (\lambda_1 + \lambda_2)s + \lambda_1\lambda_2$$

implies

$$\mathrm{tr}A = \lambda_1 + \lambda_2, \qquad \det A = \lambda_1\lambda_2.$$

Thus giving the trace and the determinant is equivalent to giving the pair of eigenvalues.

Recall that the general solution to a system $\dot{\mathbf{x}} = A\mathbf{x}$ is usually of the form $c_1e^{\lambda_1 t}\alpha_1 + c_2e^{\lambda_2 t}\alpha_2$, where λ_1, λ_2 are the eigenvalues of the matrix A and α_1, α_2 are corresponding nonzero eigenvectors. The eigenvalues by themselves usually describe most of the gross structure of the phase portrait.

There are two caveats. First, this is not necessarily the case if the eigenvalues coincide. In two dimensions, when the eigenvalues coincide one of two things happens. (1) The *complete* case. Then $A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{bmatrix}$, *every* vector is an eigenvector (for the eigenvalue $\lambda_1 = \lambda_2$), and the general solution is $e^{\lambda_1 t}\alpha$ where α is any vector. (2) The *defective* case. (This covers all the other matrices with repeated eigenvalues, so if you discover your eigenvalues are repeated and you are not diagonal, then you are defective.) Then there is (up to multiple) only one eigenvector, α_1, and the general solution is $\mathbf{x} = e^{\lambda_1 t}(c_1\alpha_1 + c_2(t\alpha_1 + \beta))$, where β is a vector such that $(A - \lambda_1 I)\beta = \alpha_1$. (Such a vector β always exists in this situation, and is unique up to addition of a multiple of α_1.)

The second caveat is that the eigenvalues may be non-real. They will then form a complex conjugate pair. The eigenvectors will also be non-real, and if α_1 is an eigenvector for λ_1 then $\alpha_2 = \overline{\alpha_1}$ is an eigenvector for $\lambda_2 = \overline{\lambda_1}$. Independent real solutions may be obtained by taking the real and imaginary parts of either $e^{\lambda_1 t}\alpha_1$ or $e^{\lambda_2 t}\alpha_2$. (These two have the same real parts and their imaginary parts differ only in sign.) This will give solutions of the general form e^{at} times a vector whose coordinates are linear combinations of $\cos(\omega t)$ and $\sin(\omega t)$, where the eigenvalues are $a \pm i\omega$.

Each of these caveats represents a failure of the eigenvalues by themselves to determine major aspects of the phase portrait. In the case of repeated eigenvalue, you get a defective node or a star node, depending upon whether you are in the defective case or the complete case. In the case of non-real eigenvalues you know you have a spiral (or a center, if the real part is zero); you know whether it is stable or unstable (look at the sign of the real part of the eigenvalues); but you do *not* know from the eigenvalues alone which way the spiral is spiraling, clockwise or counterclockwise.

31.2. **The** $(\mathrm{tr}, \det)$ **plane and structural stability.** We are now confronted with a large collection of autonomous systems, the linear two-dimensional systems $\dot{\mathbf{x}} = A\mathbf{x}$. This collection is parametrized by the four entries in the matrix. We have understood that much of the behavior of such a system is determined by two particular combinations of these four parameters, namely the trace and the determinant.

So we will consider now an entire plane with coordinates (T, D). Whenever we pick a point on this plane, we will be considering the linear autonomous systems whose matrix has trace T and determinant D.

Such a matrix is not well-defined. For given (T, D) there are always infinitely many matrices A with $\mathrm{tr}A = T$ and $\det A = D$. One example is the "companion matrix,"

$$A = \begin{bmatrix} 0 & 1 \\ -D & T \end{bmatrix}.$$

This is a particularly important example, because it represents the system corresponding to the LTI equation $\ddot{x} - T\dot{x} + Dx = 0$, via $y = \dot{x}$. (I'm sorry about the notation here. T and D are just numbers; Dx does not signify the derivative of x.)

The (T, D) plane divides into several parts according to the appearance of the phase portrait of the corresponding matrices. The important regions are as follows.

If $D < 0$, the eigenvalues are real and of opposite sign, and the phase portrait is a saddle (which is always unstable).

If $0 < D < T^2/4$, the eigenvalues are real, distinct, and of the same sign, and the phase portrait is a node, stable if $T < 0$, unstable if $T > 0$.

If $0 < T^2/4 < D$, the eigenvalues are neither real nor purely imaginary, and the phase portrait is a spiral, stable if $T < 0$, unstable if $T > 0$.

These three regions cover the whole of the (T, D) except for the curves separating them from each other, and so are them most commonly encountered and the most important cases. Suppose I have a matrix A with $(\mathrm{tr}A, \det A)$ in one of these regions. If someone kicks my matrix, so that its entries change slightly, I don't have to worry; if the change was small enough, the new matrix will be in the same region and the character of the phase portrait won't have changed very much. This is a feature known as "structural stability."

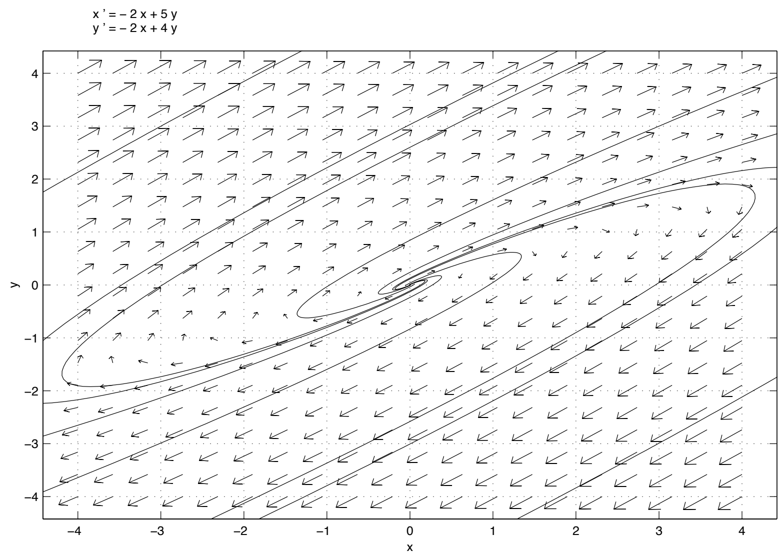
The remainder of the (T, D) plane is populated by matrices exhibiting various other phase portrait types. They are *structurally unstable*, in the sense that arbitrarily small perturbations of their entries can, and almost always will, result in a matrix with phase portrait of a different type. For example, when $0 < D$ and $T = 0$, the eigenvalues are purely imaginary, and the phase portrait is a center. But most perturbations of such a matrix will result in one whose eigenvalues have nonzero real part and hence whose phase portrait is a spiral.

31.3. **The portrait gallery.** Now for the dictionary of phase portraits. In the pictures which accompany these descriptions some elements are necessarily chosen at random. For one thing, most of the time there will be two independent eigenlines (i.e., lines through the origin made up of eigenvectors). Below, if these are real they will be the lines through $\alpha_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\alpha_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. If there is only one eigendirection (this only happens if $\lambda_1 = \lambda_2$ and is then called the "defective case") it will be the line through $\alpha_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. If they are not real, they are conjugate to each other and hence distinct. The question of how they influence the phase portrait is more complex and will not be addressed.

Name: Spiral.

Eigenvalues: Neither real nor purely imaginary: $0 \neq \mathrm{tr}^2/4 < \det$.

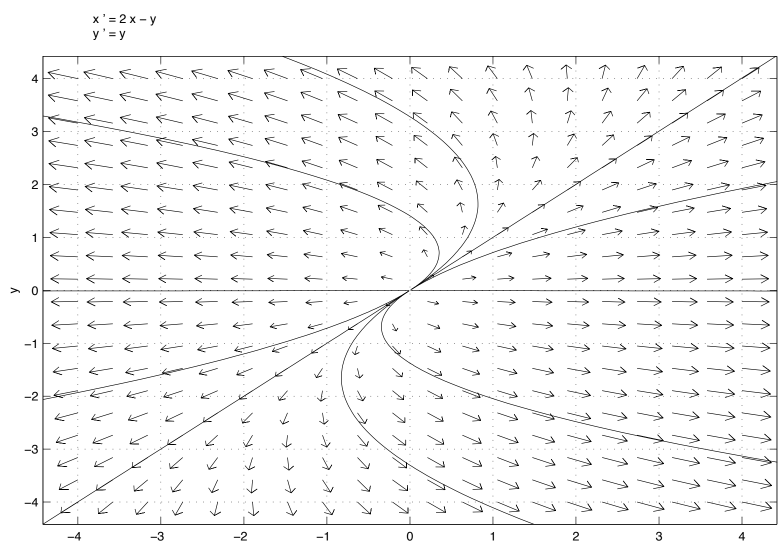
Stability: Stable if $\mathrm{tr} < 0$, unstable if $\mathrm{tr} > 0$.



Name: Node.

Eigenvalues: Real, same sign: $0 < \det < \mathrm{tr}^2/4$.

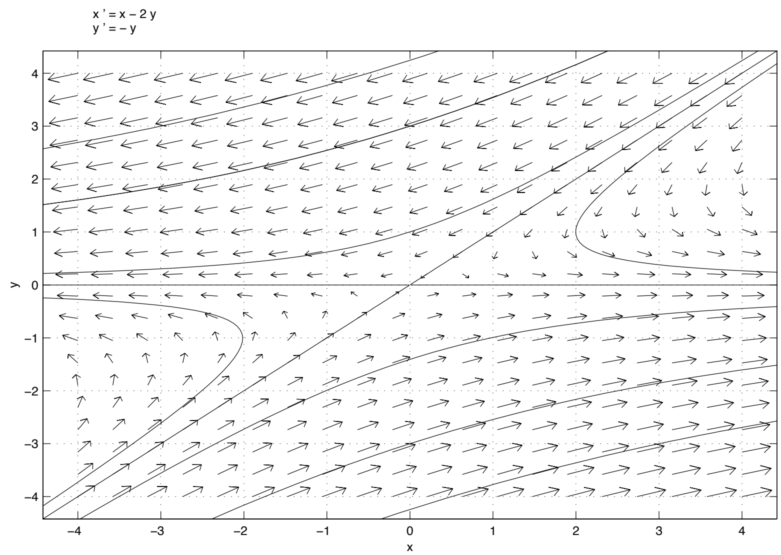
Stability: Stable if $\mathrm{tr} < 0$, unstable if $\mathrm{tr} > 0$.



Name: Saddle.

Eigenvalues: Real, opposite sign: $\det < 0$.

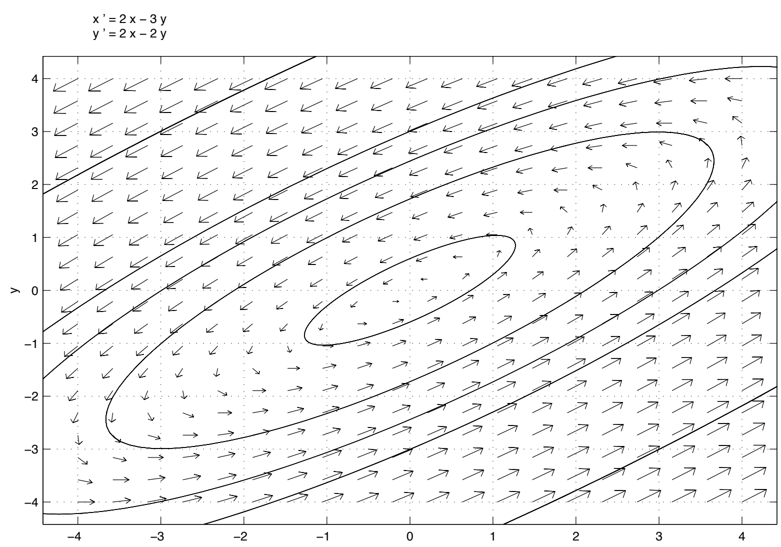
Stability: Unstable.



Name: Center.

Eigenvalues: Purely imaginary, nonzero: $\mathrm{tr} = 0, \det > 0$.

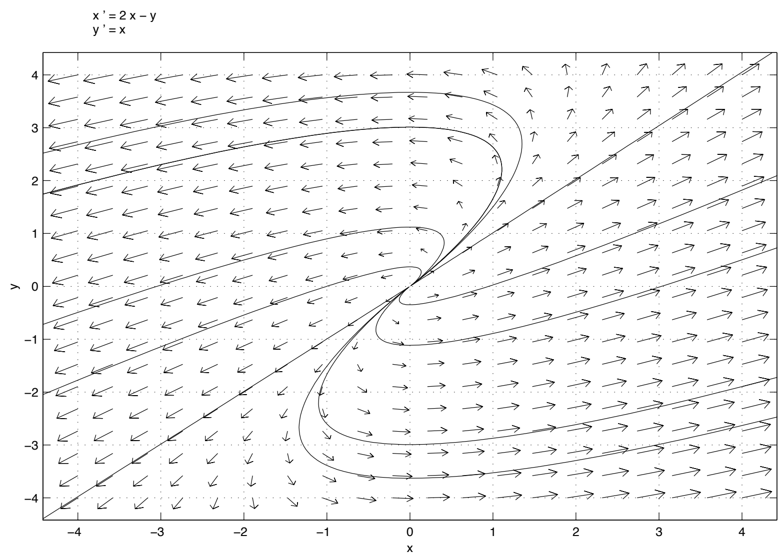
Stability: Neutrally stable.



Name: Defective Node.

Eigenvalues: Repeated (hence real) but nonzero: $\det = \mathrm{tr}^2/4 > 0$; defective.

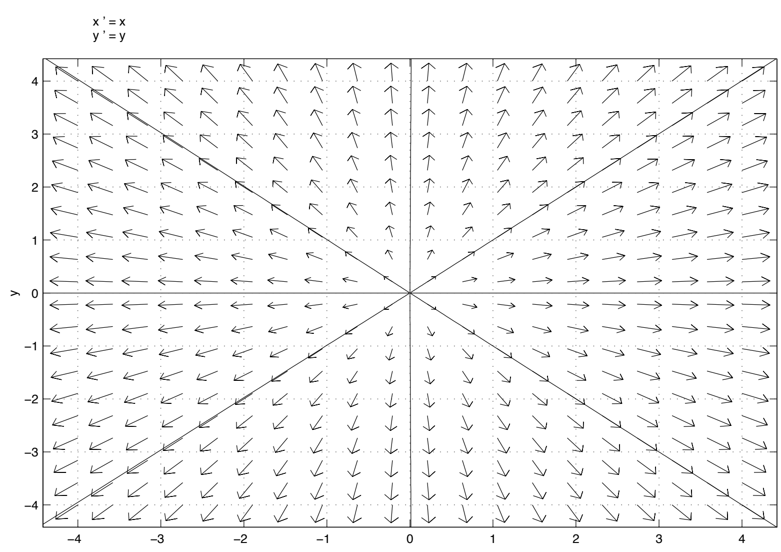
Stability: Stable if $\mathrm{tr} < 0$, unstable if $\mathrm{tr} > 0$.



Name: Star Node.

Eigenvalues: Repeated (hence real) but nonzero; complete: $\det = \mathrm{tr}^2/4 > 0$.

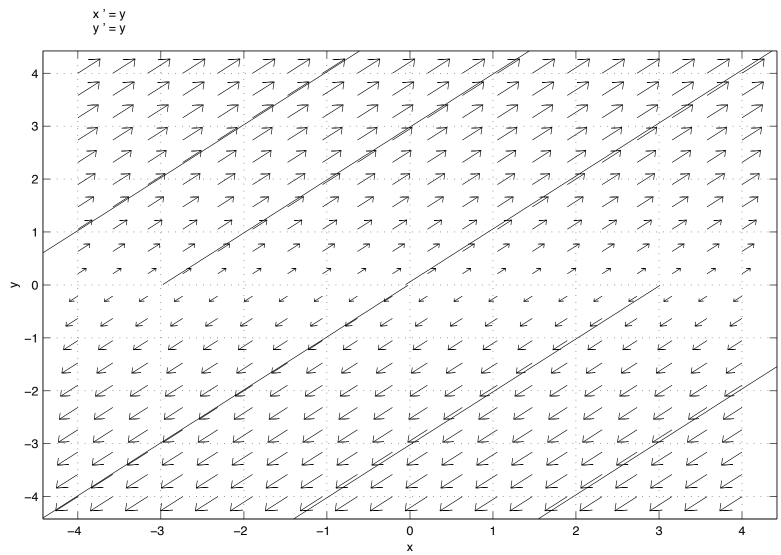
Stability: Stable if $\mathrm{tr} < 0$, unstable if $\mathrm{tr} > 0$.



Name: Degenerate: Comb.

Eigenvalues: One zero (hence both real).

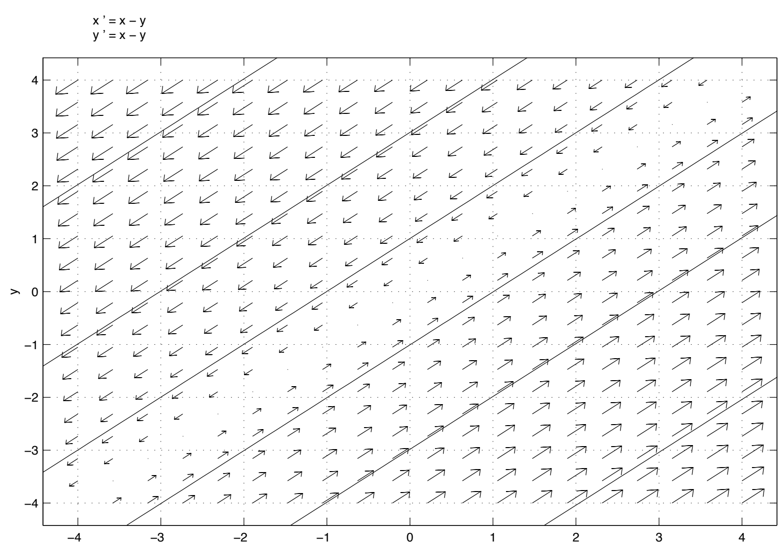
Stability: Stable if tr < 0, unstable if tr > 0.



Name: Degenerate: Parallel Lines.

Eigenvalues: Both zero: tr $=$ det $= 0$; defective.

Stability: Unstable.



Name: Degenerate: Everywhere fixed; $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Eigenvalues: Both zero: $\mathrm{tr} = \det = 0$; complete.

Stability: Neutrally stable.

(No picture; every point is a constant solution.)