30. First order systems and second order equations

30.1. **The companion system.** One of the main reasons we study first order systems is that a differential equation of any order may be replaced by an equivalent first order system. Computer ODE solvers use this principle.

To illustrate, suppose we start with a second order homogeneous LTI system,

$$\ddot{x} + b\dot{x} + cx = 0.$$

The way to replace this by a *first order* system is to introduce a new variable, say y, related to x by

$$y = \dot{x}.$$

Now we can replace $\ddot{x}$ by $\dot{y}$ in the original equation, and find a *system*:

$$(1) \qquad \begin{cases} \dot{x} & = & y \\ \dot{y} & = & -cx - by \end{cases}$$

The solution $x(t)$ of the original equation appears as the top entry in the vector-valued solution of this system.

This process works for *any* higher order equation, linear or not, provided we can express the top derivative as a function of the lower ones (and t). An nth order equation gives rise to a first order system in n variables.

The trajectories of this system represent in very explicit form many aspects of the time evolution of the original equation. You no longer have time represented by an axis, but you see the effect of time quite vividly, since the vertical direction, y, records the velocity, $y = \dot{x}$. A stable spiral, for example, reflects damped oscillation. (See the Mathlet `Damped Vibrations` for a clear visualization of this.)

The matrix for the system (1),

$$\begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix},$$

is called the *companion matrix*. These matrices constitute quite a wide range of 2×2 matrices, but they do have some special features. For example, if a companion matrix has a repeated eigenvalue then it is necessarily incomplete, since a companion matrix can never be a multiple of the identity matrix.

This association explains an apparent conflict of language: we speak of the characteristic polynomial of a second order equation—in the case

at hand it is $p(s) = s^2 + bs + c$. But we also speak of the characteristic polynomial of a matrix. Luckily (and obviously)

> The characteristic polynomial of a second order LTI operator is the same as the characteristic polynomial of the companion matrix.

30.2. **Initial value problems.** Let's do an example to illustrate this process, and see how initial values get handled. We will also use this example to illustrate some useful ideas and tricks about handling linear systems.

Suppose the second order equation is

(2) $$\ddot{x} + 3\dot{x} + 2x = 0.$$

The companion matrix is

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix},$$

so solutions to (2) are the top entries of the solutions to $\dot{\mathbf{x}} = A\mathbf{x}$.

An initial value for (2) gives us values for both x and $\dot{x}$ at some initial time, say $t = 0$. Luckily, this is exactly the data we need for an initial value for the matrix equation $\dot{\mathbf{x}} = A\mathbf{x}$: $\mathbf{x}(0) = \begin{bmatrix} x(0) \\ \dot{x}(0) \end{bmatrix}$.

Let's solve the system first, by finding the exponential e^{At}. The eigenvalues of A are the roots of the characteristic polynomial, namely $\lambda_1 = -1, \lambda_2 = -2$. (From this we know that there are two normal modes, one with an exponential decay like e^{-t}, and the other with a much faster decay like e^{-2t}. The phase portrait is a stable node.)

To find an eigenvector for λ_1, we must find a vector α_1 such that $(A - \lambda_1 I)\alpha_1 = 0$. Now

$$A - (-1)I = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}.$$

A convenient way to find a vector killed by this matrix is to take the entries from one of the rows, reverse their order and one of their signs: so for example $\alpha_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ will do nicely. The other row serves as a check; if it doesn't kill this vector then you have made a mistake somewhere. In this case it does.

Similarly, an eigenvector for λ_2 is $\alpha_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

The general solution is thus

$$(3) \qquad \mathbf{x} = ae^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + be^{-2t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

From this expression we can get a good idea of what the phase portrait looks like. There are two eigendirections, containing straight line solutions. The line through $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ contains solutions decaying like e^{-t}. (Notice that this single line contains *infinitely many* solutions: for any point $\mathbf{x}_0$ on the line there is a solution $\mathbf{x}$ with $\mathbf{x}(0) = \mathbf{x}_0$. If $\mathbf{x}_0$ is the zero vector then this is the constant solution $\mathbf{x} = \mathbf{0}$.) The line through $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ contains solutions decaying like e^{-2t}.

The general solution is a linear combination of these two. Notice that as time grows, the coefficient of $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ varies like the *square* of the coefficient of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. When time grows large, both coefficients become small, but the second becomes smaller much faster than the first. Thus the trajectory becomes asymptotic to the eigenline through $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. If you envision the node as a spider, the body of the spider is oriented along the eigenline through $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

A fundamental matrix is given by lining up the two normal modes as columns of a matrix:

$$\Phi = \begin{bmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{bmatrix}.$$

Since each column is a solution, any fundamental matrix itself is a solution to $\dot{\mathbf{x}} = A\mathbf{x}$ in the sense that

$$\dot{\Phi} = A\Phi.$$

(Remember, premultiplying Φ by A multiplies the columns of Φ by A separately.)

The exponential matrix is obtained by normalizing Φ, i.e. by postmultiplying Φ by $\Phi(0)^{-1}$ so as to obtain a fundamental matrix which is the identity at $t = 0$. Since

$$\Phi(0) = [\alpha_1 \quad \alpha_2] = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix},$$

$$\Phi(0)^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

and so

$$e^{At} = \Phi(t)\Phi(0)^{-1} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}.$$

Now any IVP for this ODE is easy to solve: $\mathbf{x} = e^{At}\mathbf{x}(0)$. For example, if $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then

$$\mathbf{x} = \begin{bmatrix} 3e^{-t} - 2e^{-2t} \\ -3e^{-t} + 4e^{-2t} \end{bmatrix}.$$

Now let's solve the original second order system, and see how the various elements of the solution match up with the work we just did. The key is always the fact that $y = \dot{x}$: $\mathbf{x} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$.

As observed, the characteristic polynomial of (2) is the same as that of A, so the eigenvalues of A are the roots, and we have two normal modes: e^{-t} and e^{-2t}. These are the exponential solutions to (2). The general solution is

$$x = ae^{-t} + be^{-2t}.$$

Note that (3) has this as top entry, and its derivative as bottom entry.

To solve general IVPs we would like to find the pair of solutions which is normalized at $t = 0$ as in Section 9. These are solutions x_1 and x_2 such that $\begin{bmatrix} x_1(0) \\ \dot{x}_1(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} x_2(0) \\ \dot{x}_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. This says exactly that we are looking for the columns of the normalized fundamental matrix e^{At}! Thus we can read off x_1 and x_2 from the top row of e^{At}:

$$x_1 = 2e^{-t} - e^{-2t}, \qquad x_2 = e^{-t} - e^{-2t}.$$

The bottom row of e^{At} is of course exactly the derivative of the top row.

The process of finding $\Phi(0)^{-1}$ is precisely the same as the process of finding the numbers a, b, c, d such that $x_1 = ae^{-t} + be^{-2t}$ and $x_2 = ce^{-t} + de^{-2t}$ form a normalized pair of solutions. If A is the companion matrix for a second order homogeneous LTI equation, then the entries in the top row of e^{At} constitute the pair of solutions of the original equation normalized at $t = 0$.