20. More on Fourier series

The Mathlet `Fourier Coefficients` displays many of the effects described in this section.

20.1. **Symmetry and Fourier series.** A function $g(t)$ is **even** if $g(t) = g(-t)$, and **odd** if $g(t) = -g(-t)$.

Fact: *Any* function $f(t)$ is a sum of an even function and an odd function, and this can be done in only one way.

The *even part* of $f(t)$ is

$$f_+(t) = \frac{f(t) + f(-t)}{2}.$$

and the *odd part* is

$$f_-(t) = \frac{f(t) - f(-t)}{2}.$$

It's easy to check that $f_+(t)$ is even, $f_-(t)$ is odd, and that

$$f(t) = f_+(t) + f_-(t).$$

We can apply this to a periodic function. We know that any periodic function $f(t)$, with period 2π, say, has a Fourier expansion of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nt) + b_n \sin(nt)).$$

If $f(t)$ is even then all the b_n's vanish and the Fourier series is simply

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt).$$

If $f(t)$ is odd then all the a_n's vanish and the Fourier series is

$$\sum_{n=1}^{\infty} b_n \sin(nt).$$

Most of the time one is faced with a function which is either even or odd. If $f(t)$ is neither even nor odd, we can still compute its Fourier series by computing the Fourier series for $f_+(t)$ and $f_-(t)$ separately and adding the results.

20.2. **Symmetry about other points.** More general symmetries are often present and useful. A function may exhibit symmetry about any fixed value of t, say $t = a$. We say that $f(t)$ is *even about* a if $f(a+t) = f(a-t)$ for all t. It is *odd about* a if $f(a+t) = -f(a-t)$. $f(t)$ is even about a if it behaves the same as you move away from a whether to the left or the right; $f(t)$ is odd about a if its values to the right of a are the negatives of its values to the left. The usual notions of even and odd refer to $a = 0$.

Suppose $f(t)$ is periodic of period 2π, and is even (about 0). $f(t)$ is then entirely determined by its values for t between 0 and π. When we focus attention on this range of values, $f(t)$ may have some *further* symmetry with respect to the midpoint $\pi/2$: it may be even about $\pi/2$ or odd about $\pi/2$, or it may be neither. For example, $\cos(nt)$ is even about $\pi/2$ exactly when n is even, and odd about $\pi/2$ exactly when n is odd. It follows that if $f(t)$ is even and even about $\pi/2$ then its Fourier series involves only *even* cosines:

$$f(t) = \frac{a_0}{2} + \sum_{n \text{ even}} a_n \cos(nt).$$

If $f(t)$ is even about 0 but odd about $\pi/2$ then its Fourier series involves only odd cosines:

$$f(t) = \sum_{n \text{ odd}} a_n \cos(nt).$$

Similarly, the odd function $\sin(nt)$ is even about $\pi/2$ exactly when n is odd, and odd about $\pi/2$ exactly when n is even. Thus if $f(t)$ is odd about 0 but even about $\pi/2$, its Fourier series involves only odd sines:

$$f(t) = \sum_{n \text{ odd}} b_n \sin(nt).$$

If it is odd about both 0 and $\pi/2$, its Fourier series involves only even sines:

$$f(t) = \sum_{n \text{ even}} a_n \sin(nt).$$

20.3. **The Gibbs effect.** The Fourier series for the odd function of period 2π with

$$F(x) = \frac{\pi - x}{2} \quad \text{for } 0 < x < \pi$$

is

$$F(x) = \sum_{k=1}^{\infty} \frac{\sin(kx)}{k}\,.$$

In Figure 12 we show the partial sum

$$F_n(x) = \sum_{k=1}^{n} \frac{\sin(kx)}{k}$$

with $n = 20$ and in Figure 13 we show it with $n = 100$. The horizontal lines of height $\pm\pi/2$ are also drawn.

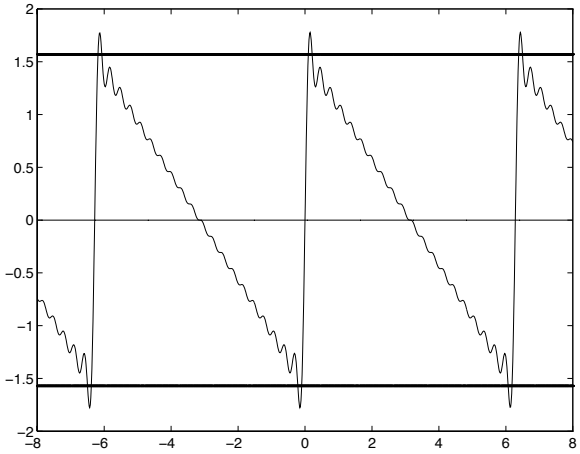

FIGURE 12. Fourier sum through $\sin(20x)$

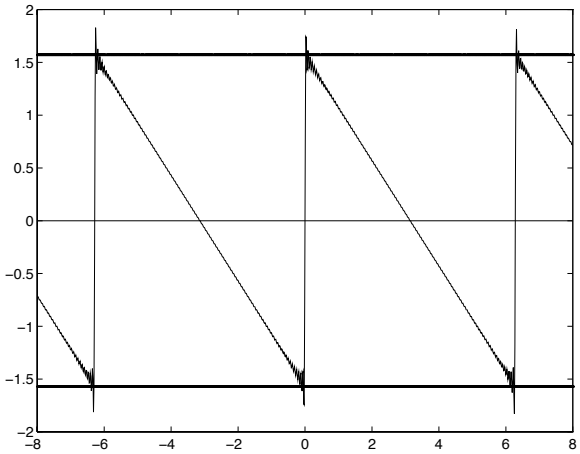

FIGURE 13. Fourier sum through $\sin(100x)$

Notice the "overshoot" near the discontinuities. If you graph $F_n(t)$ for $n = 1000$ or $n = 10^6$, you will get a similar picture. The spike near $x = 0$ will move in closer to $x = 0$, but won't get any shorter. This is the "Gibbs phenomenon." We have $F(0+) = \pi/2$, but it seems that

for any n the partial sum F_n overshoots this value by a factor of 18% or so.

A little experimentation with Matlab shows that the spike in $F_n(x)$ occurs at around $x = x_0/n$ for some value of x_0 independent of n. It turns out that we can compute the limiting value of $F_n(x_0/n)$ for *any* x_0:

Claim. For any x_0,

$$\lim_{n\to\infty} F_n\left(\frac{x_0}{n}\right) = \int_0^{x_0} \frac{\sin t}{t}\, dt.$$

To see this, rewrite the sum as

$$F_n\left(\frac{x_0}{n}\right) = \sum_{k=1}^{n} \frac{\sin(kx_0/n)}{kx_0/n} \cdot \frac{x_0}{n}.$$

Using the notation

$$f(t) = \frac{\sin t}{t}$$

this is

$$F_n\left(\frac{x_0}{n}\right) = \sum_{k=1}^{n} f\left(\frac{kx_0}{n}\right) \cdot \frac{x_0}{n}$$

You will recognize the right hand side as a Riemann sum for the function $f(t)$, between $t = 0$ and $t = x_0$. In the limit we get the integral, and this proves the claim.

To find the largest overshoot, we should look for the maximal value of $\displaystyle\int_0^{x_0} \frac{\sin t}{t}\, dt$. Figure 14 shows a graph of $\dfrac{\sin t}{t}$:

The integral hits a maximum when $x_0 = \pi$, and the later humps are smaller so it never regains this size again. We now know that

$$\lim_{n\to\infty} F_n\left(\frac{\pi}{n}\right) = \int_0^{\pi} \frac{\sin t}{t}\, dt.$$

The actual value of this definite integral can be estimated in various ways. For example, the power series for $\sin t$ is

$$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots.$$

Dividing by t and integrating term by term,

$$\int_0^{x_0} \frac{\sin t}{t}\, dt = x_0 - \frac{x_0^3}{3 \cdot 3!} + \frac{x_0^5}{5 \cdot 5!} - \cdots.$$

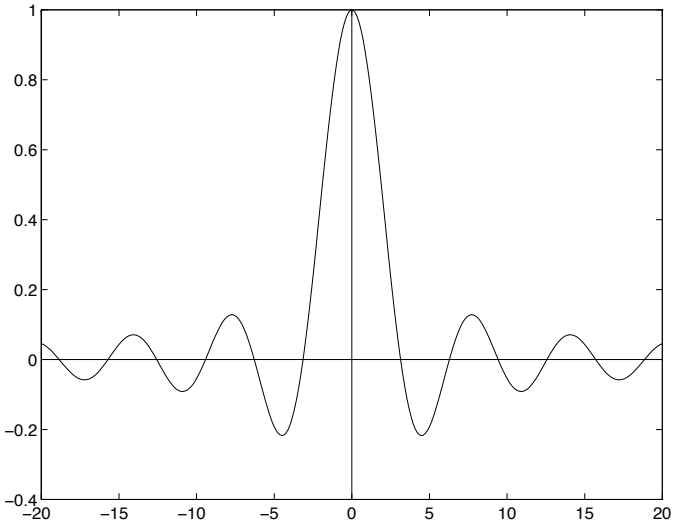

FIGURE 14. $\dfrac{\sin t}{t}$

Take $x_0 = \pi$. Pull out a factor of $\pi/2$, to compare with $F(0+) = \pi/2$:

$$\int_0^\pi \frac{\sin t}{t}\, dt = \frac{\pi}{2} \cdot G,$$

where

$$G = 2\left(1 - \frac{\pi^2}{3 \cdot 3!} + \frac{\pi^4}{5 \cdot 5!} - \cdots\right).$$

The sum converges quickly and gives

$$G = 1.17897974447216727\ldots.$$

We have found, on the graphs of the Fourier partial sums, a sequence of points which converges to the observed overshoot:

$$\left(\frac{\pi}{n}, F_n\left(\frac{\pi}{n}\right)\right) \to \left(0, (1.1789\ldots) \cdot \frac{\pi}{2}\right),$$

that is, about 18% too large. As a proportion of the *gap* between $F(0-) = -\pi/2$ and $F(0+) = +\pi/2$, this is $(G-1)/2 = 0.0894\ldots$ or about 9%. It can be shown that this is the highest overshoot.

The Gibbs overshoot occurs at every discontinuity of a piecewise continuous periodic function $F(x)$. Suppose that $F(x)$ is discontinuous at $x = a$. The overshoot comes to the same 9% of the gap, $F(a+) - F(a-)$, in every case.

Compare this effect to the basic convergence theorem for Fourier series:

Theorem. If $F(x)$ is piecewise continuous and periodic, then for any fixed number a the Fourier series evaluated at $x = a$ converges to $F(a)$ if $F(x)$ is continuous at a, and to the average $\dfrac{F(a+) + F(a-)}{2}$ in general.

The Gibbs effect does not conflict with this, because the point at which the overshoot occurs moves (it gets closer to the point of discontinuity) as n increases.

The Gibbs effect was first noticed by a British mathematician named Wilbraham in 1848, but then forgotten about till it was observed in the output of a computational machine built by the physicist A. A. Michelson (known mainly for the Michelson-Morey experiment, which proved that light moved at the same speed in every direction, despite the motion of the earth through the ether). Michelson wrote to J. Willard Gibbs, the best American physical mathematician of his age and Professor of Mathematics at Yale, who quickly wrote a paper explaining the effect.

20.4. **Fourier distance.** One can usefully regard the Fourier coefficients of a function $f(t)$ as the "coordinates" of $f(t)$ with respect to a certain coordinate system.

Imagine a vector $\mathbf{v}$ in 3-space. We can compute its x coordinate in the following way: move along the x axis till you get to the point *closest to* $\mathbf{v}$. The value of x you find yourself at *is* the x-coordinate of the vector $\mathbf{v}$.

Similarly, move about in the (x, y) plane till you get to the point which is closest to $\mathbf{v}$. This point is the orthogonal projection of $\mathbf{v}$ into the (x, y) plane, and its coordinates are the x and y coordinates of $\mathbf{v}$.

Just so, one way to think of the component $a_n \cos(nt)$ in the Fourier series for $f(t)$ is this: it is the multiple of $\cos(nt)$ which is "closest" to $f(t)$.

The "distance" between functions intended here is hinted at by the Pythagorean theorem. To find the distance between two points in Euclidean space, we take the square root of the sum of squares of differences of the coordinates. When we are dealing with functions (say on the interval between $-\pi$ and π), the analogue is

$$\boxed{\operatorname{dist}(f(t), g(t)) = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} (f(t) - g(t))^2 \, dt \right)^{1/2}}. \tag{1}$$

This number is the **root mean square distance** between $f(t)$ and $g(t)$. The fraction $1/2\pi$ is inserted so that $\text{dist}(1,0) = 1$ (rather than $\sqrt{2\pi}$) and the calculations on p. 560 of Edwards and Penney show that for $n > 0$

$$\text{dist}(\cos(nt), 0) = \frac{1}{\sqrt{2}}, \qquad \text{dist}(\sin(nt), 0) = \frac{1}{\sqrt{2}}.$$

The root mean square distance between $f(t)$ and the zero function is called the *norm* of $f(t)$, and is a kind of mean amplitude. The norm of the periodic system response is recorded as "RMS" in the Mathlets `Harmonic Frequency Response` and `Harmonic Frequency Response II`.

One may then try to approximate a function $f(t)$ by a linear combination of $\cos(nt)$'s and $\sin(nt)$'s, by adjusting the coefficients so as to minimize the "distance" from the finite Fourier sum and the function $f(t)$. The Fourier coefficients give the best possible multiples.

Here is an amazing fact. Choose coefficients a_n and b_n *randomly* to produce a function $g(t)$. Then vary one of them, say a_7, and watch the distance between $f(t)$ and this varying function $g(t)$. This distance achieves a minimum precisely when a_7 equals the coefficient of $\cos(7t)$ in the Fourier series for $f(t)$. This effect is *entirely independent* of the other coefficients you have used. You can fix up one at a time, ignoring all the others till later. You can adjust the coefficients to progressively minimize the distance to $f(t)$ in any order, and you will never have to go back and fix up your earlier work. It turns out that this is a reflection of the "orthogonality" of the $\cos(nt)$'s and $\sin(nt)$'s, expressed in the fact, presented on p. 560 of Edwards and Penney, that the integrals of products of distinct sines and cosines are always zero.

20.5. **Harmonic response.** One of the main uses of Fourier series is to express periodic system responses to general periodic signals. For example, if we drive an undamped spring with a plunger at the end of the spring, the equation is given by

$$m\ddot{x} + kx = kf(t)$$

where $f(t)$ is the position of the plunger, and the x coordinate is arranged so that $x = 0$ when the spring is relaxed and $f(t) = 0$. The natural frequency of the spring/mass system is $\omega = \sqrt{k/m}$, and dividing the equation through by m gives

$$\ddot{x} + \omega^2 x = \omega^2 f(t)\,. \tag{2}$$

This equation is illustrated in the Mathlet `Harmonic Frequency Response`.

An example is given by taking for $f(t)$ the squarewave $\mathrm{sq}(t)$, the function which is periodic of period 2π and such that

$$\mathrm{sq}(t) = \begin{cases} 1 & \text{for} \quad 0 < t < \pi \\ -1 & \text{for} \quad -\pi < t < 0 \end{cases}$$

Its Fourier series is

$$\mathrm{sq}(t) = \frac{4}{\pi}\left(\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \cdots\right). \tag{3}$$

The periodic system response to the term in the Fourier series for $\omega^2\mathrm{sq}(t)$

$$\frac{4\omega^2}{\pi n}\sin(nt)$$

(where n is an odd integer) is, by the Exponential Reponse Formula (10.10),

$$\frac{4\omega^2}{\pi n}\cdot\frac{\sin(nt)}{\omega^2 - n^2}\,.$$

Thus the periodic system response to $f(t) = \mathrm{sq}(t)$ is given by the Fourier series

$$x_p(t) = \frac{4\omega^2}{\pi}\left(\frac{\sin t}{\omega^2 - 1} + \frac{\sin(3t)}{3(\omega^2 - 9)} + \cdots\right) \tag{4}$$

as long as ω isn't one of the frequencies of the Fourier modes of the signal, i.e. the odd integers.

This expression explains important general features of the periodic solution. When the natural frequency of the system, ω, is near to one of the frequencies present in the Fourier series for the signal (odd integers in this example), the periodic system response x_p is dominated by the near resonant response to that mode in the signal. When ω is slightly larger than $(2k+1)$ the system response is in phase; as ω decreases though the value $(2k+1)$, the system passes through resonance with the signal (and when $\omega = 2k+1$ there is *no* periodic solution), and comes out on the other side in anti-phase.

In this example, and many others, however, the *same* solution can be obtained quite easily using standard methods of linear ODEs, using some simple features of the solution. These features can be seen directly from the equation, but from our present perspective it's easier to see them from (4). They are:

$$x_p(0) = 0\,, \qquad x_p(\pi) = 0\,.$$

I claim that as long as ω isn't an integer, (2) has just one solution with these properties. That solution is given as a Fourier series by (4), but we can write it out differently using our older methods.

In the interval $[0, \pi]$, the equation is simply

$$\ddot{x} + \omega^2 x = \omega^2 \,.$$

We know very well how to solve this! A particular solution is given by a constant, namely 1, and the general solution of the homogeneous equation is given by $a\cos(\omega t) + b\sin(\omega t)$. So

$$x_p = 1 + a\cos(\omega t) + b\sin(\omega t)$$

for some constants a, b.

Substituting $t = 0$ gives $a = -1$, so

$$x_p = 1 - \cos(\omega t) + b\sin(\omega t)\,, \qquad 0 < t < \pi. \tag{5}$$

Substituting $t = \pi$ gives the value for b, depending upon ω:

$$b = \frac{\cos(\pi\omega) - 1}{\sin(\pi\omega)}\,.$$

In the interval $[-\pi, 0]$, the complete signal is $-\omega^2$, so exactly the same calculation gives the negative of the function just written down. Therefore the solution x_p is the *odd* function of period 2π extending

$$x_p = 1 - \cos(\omega t) + \left(\frac{\cos(\pi\omega) - 1}{\sin(\pi\omega)}\right)\sin(\omega t)\,, \qquad 0 < t < \pi\,. \tag{6}$$

The Fourier series of this function is given by (4), but I for one would never have guessed that the expression (4) summed up to such a simple function.

Let's finish up our analysis of this example by thinking about the situation in which the natural frequency ω equals the angular frequency of one of the potential Fourier components of the signal—i.e., an integer, in this case.

In case ω is an even integer, the expression for b is indeterminate since both numerator and denominator are zero. However, in this case the function $x_p = 1 - \cos(\omega t)$ already satisfies $x_p(\pi) = 0$, so we can (and must!) take $b = 0$. Thus x_p is the odd extension of $1 - \cos(\omega t)$. In this case, however, notice that this is not the only periodic solution; indeed, in this case *all* solutions are periodic, since the general solution is (writing $\omega = 2k$)

$$x_p + c_1\cos(2kt) + c_2\sin(2kt)$$

and all these are periodic of period 2π.

In case ω is an odd integer, $\omega = 2k+1$, there are no periodic solutions; the system is in resonance with the Fourier mode $\sin((2k+1)t)$ present in the signal. We can't solve for the constant b; the zero in its denominator is not canceled by a zero in its numerator. It is not hard to write down a particular solution in this case too, using the Resonance Exponential Response Formula, Section 14.

We have used the undamped harmonic oscillator for this example, but the same methods work in the presence of damping. In that case it is much easier to use the complex form of the Fourier series (Section 20.6 below) since the denominator in the Exponential Response Formula is no longer real.

20.6. **Complex Fourier series.** With all the sines and cosines in the Fourier series, there must be a complex exponential expression for it. There is, and it looks like this:

$$\boxed{f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}} \tag{7}$$

The power and convenience we have come to appreciate in the complex exponential is at work here too, making computations much easier. Complex Fourier series are illustrated in the Mathlet `Fourier Coefficients: Complex with Sound`.

To obtain an integral expression for one of these coefficients, say c_m, the first step is to multiply the expression (7) by e^{-imt} and integrate:

$$\int_{-\pi}^{\pi} f(t)e^{-imt}dt = \sum_{n=-\infty}^{\infty} c_n \int_{-\pi}^{\pi} e^{i(n-m)t}dt \tag{8}$$

Now

$$\int_{-\pi}^{\pi} e^{i(n-m)t}dt = \begin{cases} 2\pi & \text{if } m = n \\ \left.\dfrac{e^{i(n-m)t}}{i(n-m)}\right|_{-\pi}^{\pi} = 0 & \text{if } m \neq n, \end{cases}.$$

The top case holds because then the integrand is the constant function 1. The second case follows from $e^{i(n-m)\pi} = (-1)^{n-m} = e^{i(n-m)(-\pi)}$. Thus only one term in (8) is nonzero, and we conclude that

$$\boxed{c_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-imt}\,dt} \tag{9}$$

This works perfectly well even if $f(t)$ is complex valued. When $f(t)$ is in fact real valued, so that $\overline{f(t)} = f(t)$, (9) implies first that c_0 is real; it's the average value of $f(t)$, that is, in the older notation for Fourier coefficients, $c_0 = a_0/2$. Also, $c_{-n} = \overline{c_n}$ because

$$c_{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-i(-n)t}dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{f(t)}e^{int}dt = \overline{c_n}.$$

Since also $e^{-int} = \overline{e^{int}}$, the nth and $(-n)$th terms in the sum (7) are conjugate to each other. We will group them together. The numbers will come out nicely if we choose to write

$$c_n = (a_n - ib_n)/2 \tag{10}$$

with a_n and b_n real. Then $c_{-n} = (a_n + ib_n)/2$, and we compute that

$$c_n e^{int} + c_{-n}e^{-int} = 2\mathrm{Re}\left(c_n e^{int}\right) = a_n\cos(nt) + b_n\sin(nt).$$

(I told you the numbers would work out well, didn't I?) The series (7) then becomes the usual series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos(nt) + b_n\sin(nt)\right).$$

Moreover, taking real and imaginary parts of the integral (9) (and continuing to assume $f(t)$ is real valued) we get the usual formulas

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt, \qquad b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)dt.$$