15. Natural frequency and damping ratio

There is a standard, and useful, normalization of the second order homogeneous linear constant coefficient ODE

$$m\ddot{x} + b\dot{x} + kx = 0$$

under the assumption that both the "mass" m and the "spring constant" k are positive. It is illustrated in the Mathlet `Damping Ratio`.

In the absence of a damping term, the ratio k/m would be the square of the angular frequency of a solution, so we will write $k/m = \omega_n^2$ with $\omega_n > 0$, and call ω_n the **natural angular frequency** of the system.

Divide the equation through by m: $\ddot{x} + (b/m)\dot{x} + \omega_n^2 x = 0$. Critical damping occurs when the coefficient of $\dot{x}$ is $2\omega_n$. The **damping ratio** ζ is the ratio of b/m to the critical damping constant: $\zeta = (b/m)/(2\omega_n)$. The ODE then has the form

$$\boxed{\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0} \tag{1}$$

Note that if x has dimensions of cm and t of sec, then ω_n had dimensions $\sec^{-1}$, and the damping ratio ζ is "dimensionless." This implies that it is a number which is the same no matter what units of distance or time are chosen. Critical damping occurs precisely when $\zeta = 1$: then the characteristic polynomial has a repeated root: $p(s) = (s + \omega_n)^2$.

In general the characteristic polynomial is $s^2 + 2\zeta\omega_n s + \omega_n^2$, and it has as roots

$$-\zeta\omega_n \pm \sqrt{\zeta^2\omega_n^2 - \omega_n^2} = \omega_n(-\zeta \pm \sqrt{\zeta^2 - 1}).$$

These are real when $|\zeta| \geq 1$, equal when $\zeta = \pm 1$, and nonreal when $|\zeta| < 1$. When $|\zeta| \leq 1$, the roots are

$$-\zeta\omega_n \pm i\omega_d$$

where

$$\omega_d = \sqrt{1 - \zeta^2}\,\omega_n \tag{2}$$

is the **damped angular frequency** of the system. Recall that if r_1 and r_2 are the roots of the quadratic $s^2 + bs + c$ then $r_1 r_2 = c$ and $r_1 + r_2 = -b$. In our case, the roots are complex conjugates, so their product is the square of their modulus, which is thus ω^n. Their sum is twice their common real part, which is thus $-\zeta\omega_n$. The real part of a complex number z is $|z|\cos(\mathrm{Arg}(z))$, so we find that the arguments of the roots are $\pm\theta$, where $-\zeta = \cos\theta$. Note that the presence of a damping term decreases the frequency of a solution to the undamped equation—the natural frequency ω_n—by the factor $\sqrt{1-\zeta^2}$. The general solution is

$$x = Ae^{-\zeta\omega_n t}\cos(\omega_d t - \phi) \tag{3}$$

Suppose we have such a system, but don't know the values of ω_n or ζ. At least when the system is underdamped, we can discover them by a couple of simple measurements of the system response. Let's displace the mass and watch it vibrate freely. If the mass oscillates, we are in the underdamped case.

We can find ω_d by measuring the times at which x achieves its maxima. These occur when the derivative vanishes, and

$$\dot{x} = Ae^{-\zeta\omega_n t}\left(-\zeta\omega_n\cos(\omega_d t - \phi) - \omega_d\sin(\omega_d t - \phi)\right).$$

The factor in parentheses is sinusoidal with angular frequency ω_d, so successive zeros are separated from each other by a time lapse of π/ω_d. If t_1 and t_2 are the times of neighboring maxima of x (which occur at every other extremum) then $t_2 - t_1 = 2\pi/\omega_d$, so we have discovered the damped natural frequency:

$$\omega_d = \frac{2\pi}{t_2 - t_1}. \tag{4}$$

Here are two ways to measure the damping ratio ζ.

1. We can measure the ratio of the value of x at two successive maxima. Write $x_1 = x(t_1)$ and $x_2 = x(t_2)$. The difference of their natural logarithms is the **logarithmic decrement**:

$$\Delta = \ln x_1 - \ln x_2 = \ln\left(\frac{x_1}{x_2}\right).$$

Then

$$x_2 = e^{-\Delta}x_1.$$

The logarithmic decrement turns out to depend only on the damping ratio, and to determine the damping ratio. To see this, note that the values of $\cos(\omega_d t - \phi)$ at two points of time differing by $2\pi/\omega_d$ are equal. Using (3) we find

$$\frac{x_1}{x_2} = \frac{e^{-\zeta\omega_n t_1}}{e^{-\zeta\omega_n t_2}} = e^{\zeta\omega_n(t_2-t_1)}.$$

Thus, using (4) and (2),

$$\Delta = \ln\left(\frac{x_1}{x_2}\right) = \zeta\omega_n(t_2 - t_1) = \zeta\omega_n\,\frac{2\pi}{\omega_d} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}.$$

From the quantities ω_d and Δ, which are directly measurable characteristics of the unforced system response, we can calculate the system parameters ω_n and ζ:

$$(5) \qquad \zeta = \frac{\Delta/2\pi}{\sqrt{1+(\Delta/2\pi)^2}}\,, \qquad \omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} = \sqrt{1+\left(\frac{\Delta}{2\pi}\right)^2}\,\omega_d\,.$$

2. Another way to determine the damping ratio, at least if it's reasonably small, is to count the number of cycles it takes for the system response to decay to half its original amplitude. Write $n_{1/2}$ for this number. We know that the amplitude has decayed to half its value at $t=0$ when $t=t_{1/2}$, where

$$e^{-\zeta\omega_n t_{1/2}} = 1/2$$

or $\zeta\omega_n t_{1/2} = \ln 2$. The pseudoperiod is $2\pi/\omega_d$, so

$$\frac{2\pi}{\omega_d} n_{1/2} = t_{1/2} = \frac{\ln 2}{\zeta\omega_n}$$

or

$$\zeta = \frac{\ln 2}{2\pi}\frac{\omega_d}{\omega_n}\frac{1}{n_{1/2}}\,.$$

When ζ is small, ω_d/ω_n is quite close to 1, and $\dfrac{\ln 2}{2\pi} \simeq 0.110$. So to a good approximation

$$\zeta \simeq \frac{0.11}{n_{1/2}}\,.$$