18.03 Practice Problems – Laplace Transform

Rules for the Laplace transform

Definition:	$\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}\,dt$ for $\mathrm{Re}(s) \gg 0$.
Linearity:	$\mathcal{L}[af + bg] = aF + bG$.
s-shift rule:	$\mathcal{L}[e^{rt}f(t)] = F(s-r)$.
t-shift rule:	$\mathcal{L}[f(t-a)] = e^{-as}F(s)$ if $a \geq 0$ and $f(t) = 0$ for $t < 0$.
t-derivative rule:	$\mathcal{L}[f'(t)] = sF(s) - f(0^-)$.
$\mathcal{L}^{-1}$:	$F(s)$ essentially determines $f(t)$ for $t > 0$.

Formulas for the Laplace transform

$$\mathcal{L}[1] = \frac{1}{s} \quad , \qquad \mathcal{L}[\delta(t-a)] = e^{-as}$$

$$\mathcal{L}[e^{rt}] = \frac{1}{s-r} \quad , \qquad \mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}[\cos(\omega t)] = \frac{s}{s^2+\omega^2} \quad , \qquad \mathcal{L}[\sin(\omega t)] = \frac{\omega}{s^2+\omega^2}$$

1. Use the rules and formulas to find the Laplace transform of $e^{-t}(t^2+1)$.

2. Let $f(t) = e^{-t}\cos(3t)$.

(a) From the rules and tables, what is $F(s) = \mathcal{L}[f(t)]$?

(b) Compute the derivative $f'(t)$ and its Laplace transform. Verify the t-derivative rule in this case.

3. Use the Laplace transform to find the unit impulse response and the unit step response of the operator $D + 2I$.

4. Find the inverse Laplace transform for each of the following.

$$\frac{2s+1}{s^2+9} \quad , \qquad \frac{s^3+2}{s^3(s+2)} .$$

5. Use the Laplace transform to find the solution to $\dot{x} + 2x = t^2$ with initial condition $x(0) = 1$.