18.03 Practice Problems Solutions – Convolution

1. $t^n * 1 = \displaystyle\int_0^t \tau^n \, d\tau = \frac{t^{n+1}}{n+1}$.

$1 * t^n = \displaystyle\int_0^t (t-\tau)^n \, d\tau$. Substitute $y = t - \tau$, $dy = -d\tau$, $1 * t^n = \displaystyle\int_t^0 y^n (-d\tau) = \int_0^t y^n \, d\tau = \frac{t^{n+1}}{n+1}$.

2. $q(t) * 1 = \displaystyle\int_0^t q(t) \, d\tau$ is the cumulative total deposits. In a cumulative total, the contribution made at time τ neither increases nor decreases as time moves on; the "weight function" is 1.

$q(t) * e^{It} = \displaystyle\int_0^t q(\tau) e^{I(t-\tau)} \, d\tau$ is the money in my account arising from my deposits at rate $q(t)$ between time 0 and time t. It is the solution of the LTI equation $\dot{x} - Ix = q(t)$ with rest initial conditions. The weight function of the operator $D - I$ (sorry, the I here is the interest rate, and the identity operator is going un-denoted) is $u(t)e^{It}$: this is the growth of a single dollar deposited at time $t = 0$.

3. $w(t) = u(t)e^{-kt}$. $q(t) * w(t) = 1 * e^{-kt} = \displaystyle\int_0^t 1e^{-k(t-\tau)} \, d\tau = e^{-kt} \int_0^t e^{k\tau} \, d\tau = e^{-kt} \left. \frac{e^{k\tau}}{k} \right|_0^t$

$= e^{-kt} \left(\dfrac{e^{k\tau} - 1}{k} \right) = \dfrac{1 - e^{-kt}}{k}$ This is indeed the desired solution. It's the unit step response!

4. $\delta(t-a) * g(t) = \displaystyle\int_0^t \delta(\tau - a) g(t - \tau) \, d\tau$. Remember that $\displaystyle\int_{<a}^{>a} \delta(\tau - a) f(\tau) \, d\tau = f(a)$. So $\delta(t-a) * g(t) = g(t-a)$.

5. (a) Use complex replacement and ERF, or remember $x_p = \dfrac{\cos(t)}{4-1} = \dfrac{\cos(t)}{3}$. The general solution is $x = \frac{1}{3}\cos(t) + a\cos(2t) + b\sin(2t)$. $0 = x(0) = \frac{1}{3} + a$, so $a = -\frac{1}{3}$. $\dot{x} = -\frac{1}{3}\sin t - 2a\sin(2t) + 2b\cos(2t)$, $0 = \dot{x}(0) = 2b$, so $b = 0$: $x = \frac{1}{3}(\cos(t) - \cos(2t))$.

(b) For $t > 0$, the unit impulse response satisfies $\ddot{w} + 4w = 0$, $w(0+) = 0$, $\dot{w}(0+) = 1$. $w(t) = a\cos(2t) + b\sin(2t)$, and $w(0+) = 0$ forces $a = 0$. Then $\dot{w} = 2b\cos(2t)$, so $1 = \dot{w}(0+) = 2b$ and $b = \frac{1}{2}$: $w(t) = u(t)\frac{1}{2}\sin(2t)$.

So $(q * w)(t) = \displaystyle\int_0^t \cos(\tau) \frac{\sin(2(t-\tau))}{2} \, d\tau$. This is a pain in the neck. It might simplify a little if we use commutativity: $(w * q)(t) = \dfrac{1}{2}\displaystyle\int_0^t \sin(2\tau)\cos(t-\tau) \, d\tau$. Now $\frac{1}{2}\sin(2\tau)\cos(t-\tau) = \sin(\tau)\cos(\tau)(\cos(t)\cos(\tau) + \sin(t)\sin(\tau)) = \cos(t)\cos^2(\tau)\sin(\tau) + \sin(t)\sin^2(\tau)\cos(\tau)$, which you can integrate using the substitution $y = \cos(\tau)$ for the first term and $y = \sin(\tau)$ for the second: $(w * q)(t) = -\cos(t)\dfrac{\cos^3(t) - 1}{3} - \sin(t)\dfrac{\sin^3(t)}{3} = \dfrac{1}{3}\left(\cos(t) - (\cos^4(t) - \sin^4(t))\right)$. Now $\cos^4(t) - \sin^4(t) = (\cos^2(t) - \sin^2(t))(\cos^2(t) + \sin^2(t)) = \cos(2t)$, so we find again $\frac{1}{3}(\cos(t) - \cos(2t))$.