18.03 Practice Problems: Convolution

Convolution product: The convolution product of two functions $f(t)$ and $g(t)$ is
$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau)\, d\tau.$$

Suppose that $w(t)$ is the unit impulse response (or weight function) for the operator $p(D)$. Then the solution to $p(D)x = q(t)$ with rest initial conditions is given (for $t > 0$) by $q(t) * w(t)$.

1. Compute $t^n * 1$ and $1 * t^n$.

2. If $q(t)$ is my rate of savings, how can we interpret $q(t) * 1$? If my bank's interest rate is I per year, write the integral for $q(t) * e^{It}$ and interpret it.

3. Check that $q(t) * w(t)$ is indeed the solution to $p(D)x = q(t)$ with rest initial conditions in case $p(D)x = \dot{x} + kx$ and $q(t) = 1$.

4. Suppose $g(t) = 0$ for $t < 0$ and let $a > 0$. Compute $\delta(t - a) * g(t)$.

5. (a) Solve $\ddot{x} + 4x = \cos(t)$ with rest initial conditions using the exponential response formula or formulas derived from it.

(b) Now do this by convolving with the unit impulse response.