18.03 Practice Problems on Fourier Series

For definitions and conventions, see overleaf.

1. What is the Fourier series for $1 + \sin^2 t$?

2. Graph the function $f(t)$ which is even, periodic of period 2π, and such that $f(t) = 2$ for $0 < t < \frac{\pi}{2}$ and $f(t) = 0$ for $\frac{\pi}{2} < t < \pi$. Is the function even, odd, or neither?

Find its Fourier series in two ways:

(a) Use parity if possible to see that some coefficients are zero. Then use the integral expressions for the remaining Fourier coefficients.

(b) Express $f(t)$ in terms of $\mathrm{sq}(t)$, substitute the Fourier series for $\mathrm{sq}(t)$ and use some trig identities.

3. Graph the function $h(t)$ which is odd and periodic of period 2π and such that $h(t) = t$ for $0 < t < \frac{\pi}{2}$ and $h(t) = \pi - t$ for $\frac{\pi}{2} < t < \pi$. What is its average value? Observe that $h'(t) = f(t) - 1$, where $f(t)$ is the function studied in **Problem 2**.

Use these observations to find its Fourier series.

4. Explain why any function $F(x)$ is a sum of an even function and an odd function in just one way. Hint: The function $F_+(x)$ defined as $F_+(x) = \dfrac{F(x) + F(-x)}{2}$ is even.

What is the even part of e^x? What is the odd part?

Fourier series facts and conventions

Suppose that $f(t)$ a periodic function for which 2π is a period (so that $f(t+2\pi) = f(t)$). We will assume that $f(t)$ is piecewise continuous and that

$$f(a) = \frac{f(a-) + f(a+)}{2}$$

at points of discontinuity.

Then there is exactly one sequence of numbers $a_0, a_1, a_2, \ldots, b_1, b_2, \ldots$, such that for all t:

$$\begin{aligned} f(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + \sum_{n=1}^{\infty} b_n \sin(nt) \\ &= \frac{a_0}{2} + a_1 \cos(t) + a_2 \cos(2t) + \cdots + b_1 \sin(t) + b_2 \sin(2t) + \cdots \end{aligned}$$

This expansion is called the *Fourier series* for $f(t)$, and the numbers from this sequence are defined to be the *Fourier coefficients* of $f(t)$ (regarded as a function of period 2π).

The Fourier coefficients of such a function $f(t)$ can be calculated directly by using integral formulas

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt)\, dt\,, \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt)\, dt,$$

The standard squarewave $sq(t)$ is defined to be the odd function $\mathrm{sq}(t)$ of period 2π such that $\mathrm{sq}(t) = 1$ for $0 < t < \pi$ and $\mathrm{sq}(0) = \mathrm{sq}(\pi) = 0$. Its Fourier series is

$$\mathrm{sq}(t) = \frac{4}{\pi} \left(\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \cdots \right) = \frac{4}{\pi} \sum_{k \text{ odd}} \frac{\sin(kt)}{k}.$$