

18.03 Practice Problem Solutions

Frequency response

1. Start with $b = 0$. Vary ω . What do you observe? Write down the differential equation, make the complex replacement, and compute the complex gain $H(\omega)$, the gain $h(\omega)$, and the phase lag $\phi(\omega)$. Careful! The gain is an absolute value.

This is the harmonic oscillator. With $y_{\text{cx}} = e^{i\omega t}$, the ERF gives $z_p = \frac{e^{i\omega t}}{k - m\omega^2}$. The complex gain is $H(\omega) = \frac{1}{k - m\omega^2}$, $h(\omega) = |k - m\omega^2|^{-1/2}$. The phase lag is 0 if $\omega < \omega_n$, where $\omega_n = \sqrt{k/m}$, and π if $\omega > \omega_n$.

2. Now take $\omega = 0$. Vary the system parameters and observe what the system response is. Any observations about it? Now compute it, and verify whatever you observed. Which system parameters does it depend upon? Does this make sense physically?

With $\omega = 0$, the input signal is the constant function 1. The steady state solution is $1/k$. It's independent of m and b , since those parts of the system only come into play if the mass is moving.

3. Take $m = 1$, $b = \frac{1}{4}$, and $k = 2$ (or other values, if you prefer!) Observe the variation of the system response as ω is varied. Any observations? Invoke [Bode Plots] and run the $[\omega]$ slider again. Get used to the way the amplitude and the phase lag are represented. Then invoke the [Nyquist Plot]. This shows the complex gain, and its trajectory in the complex plane as ω varies. Watch it as you vary ω . What can you say about the yellow and green elements?

Any observations involving the modulus and argument of complex numbers are good.

4. Compute $H(\omega)$, $h(\omega)$, and $\tan \phi(\omega)$, in general, in terms of m , b and k .

$$H(\omega) = \frac{1}{p(i\omega)} = \frac{1}{(k - m\omega^2) + ib\omega}. \quad h(\omega) = \frac{1}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}}. \quad \tan \phi(\omega) = \frac{b\omega}{m\omega^2 - k}.$$

5. The natural frequency of this system is defined to be the natural frequency of the system if the damping term is omitted: $\omega_n = \sqrt{k/m}$. What is the phase lag when $\omega = \omega_n$? Make some observations, and then a computation.

If $\omega = \omega_n$, the real part of $p(i\omega)$ vanishes, so the phase lag is $\pm \frac{\pi}{2}$. In this case $p(i\omega) = ib\omega$, and we are always assuming $b \geq 0$ and $\omega \geq 0$, so the imaginary part is positive. This implies that $\tan \phi(\omega) = \frac{\pi}{2}$.

6. You might expect that the natural frequency is the frequency at which the gain is maximal; this certainly happens in the absence of damping! Does it seem to happen experimentally? Is there always (i.e. for all values of m , b , k) a positive value ω_r of ω at which the gain is maximal? Now go ahead and compute what the resonant frequency ω_r is, in terms of m , b and k . (Hint: minimize the square of the denominator in the expression for $h(\omega)$.)

$\frac{d}{d\omega} ((k - m\omega^2)^2 + b^2\omega^2) = 2(k - m\omega^2)(-2m\omega) + 2b^2\omega$. This vanishes if $\omega = \sqrt{\frac{k}{m} - \frac{b^2}{2m^2}}$ or if $\omega = 0$. The first root occurs only if $\frac{k}{m} > \frac{b^2}{2m^2}$. In the other case there is no resonant peak.