PROBLEM SET 10 (DUE IN LECTURE ON NOV 19 (THURSDAY))

(All Theorem and Exercise numbers are references to the textbook by Apostol; for instance "Exercise 1.15-3" means Exercise 3 in section 1.15.)

Problem 1. Compute the following limits.

(a) $\lim_{x\to 0} \dfrac{1-\cos x}{x^2}$

(b) $\lim_{x\to 0} \dfrac{\log(1+x)}{e^{2x}-1}$

(c) $\lim_{x\to 0} \left(\dfrac{(1+x)^{1/x}}{e}\right)^{1/x}$

(d) $\lim_{x\to 0} \dfrac{\log(\cos(2x))}{\log(\cos(3x))}$

Problem 2. Let $f(x) = \tan^{-1}(x)$. Show that for any nonnegative integer n, the $(2n+1)$th Taylor polynomial of f at 0 is

$$T_{2n+1}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{2k+1} x^{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1}.$$

(Hint: first compute the Taylor polynomials of $f'(x) = \frac{1}{1+x^2}$ by using the fact that $(1+x^2)f'(x) = 1$.)

Problem 3. Do Exercise 7.13-15.

Problem 4. Suppose that f is defined in some interval containing 0 and satisfies

$$f(x) = 1 + x + o(x) \text{ as } x \to 0.$$

Prove that

$$\lim_{x\to 0} f(x)^{1/x} = e.$$

(Hint: Show that $\log x = x - 1 + o(x-1)$ as $x \to 1$.)