18.014 Final Solutions

Sam Elder

December 18, 2015

Problem 1. Let $f : [-\pi, \pi] \to \mathbb{R}$ be defined by $f(x) = \sqrt{1 - \cos x}$.

Part 1.1. Is f differentiable?

Answer. No, f is not differentiable at $x = 0$.

Solution. The chain rule shows that f is differentiable except when $1 - \cos x = 0$, which only holds for $x = 0$ within the range $[-\pi, \pi]$. To prove that f is not differentiable at $x = 0$, we must show that the limit

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x} = \lim_{x\to 0} \frac{\sqrt{1-\cos x} - \sqrt{1 - \cos 0}}{x} = \lim_{x\to 0} \frac{\sqrt{1-\cos x}}{x}$$

does not exist. To do so, we examine the left and right limits:

$$\lim_{x\to 0^+} \frac{\sqrt{1-\cos x}}{x} = \lim_{x\to 0} \sqrt{\frac{1-\cos x}{x^2}} = \sqrt{\lim_{x\to 0} \frac{1-\cos x}{x^2}} = \frac{1}{\sqrt{2}}$$

as we have seen many times before. For the left, limit, however,

$$\lim_{x\to 0^-} \frac{\sqrt{1-\cos x}}{x} = \lim_{x\to 0} -\sqrt{\frac{1-\cos x}{x^2}} = -\sqrt{\lim_{x\to 0} \frac{1-\cos x}{x^2}} = -\frac{1}{\sqrt{2}}.$$

Since these limits are not equal, the derivative does not exist. (Locally, it will look like $|x|/\sqrt{2}$.) □

Part 1.2. Is f integrable?

Answer. Yes.

Solution 1. As a composition of continuous functions, f is continuous, and therefore, integrable. □

Solution 2. We claim that f is monotone decreasing on $[-\pi, 0]$ and monotone increasing on $[0, \pi]$. Indeed, since f is continuous on both of these intervals, we merely need to compute its derivative. By the chain rule, $f'(x) = \frac{\sin x}{\sqrt{1-\cos x}}$. Since $\sin x < 0$ on $(-\pi, 0)$ and $\sin x > 0$ on $(0, \pi)$, f is indeed monotone decreasing on $[-\pi, 0]$ and monotone increasing on $[0, \pi]$. Therefore, f is integrable on $[-\pi, \pi]$, as desired. □

Problem 2.

Part 2.1. Let $x \neq 1$ be a real number. Prove by induction that for any positive integer n,

$$(1 + x)(1 + x^2)(1 + x^4)\cdots(1 + x^{2^n}) = \frac{1 - x^{2^{n+1}}}{1 - x}.$$

(Here the exponents in the factors on the LHS are powers of 2.)

Solution. We induct on n.

Base Case. $n = 1$. (You could also use $n = 0$.) Then

$$(1+x)(1+x^2) = 1 + x + x^2 + x^3$$
$$(1-x)(1+x)(1+x^2) = 1 - x + x - x^2 + x^2 - x^3 + x^3 - x^4 = 1 - x^4$$
$$(1+x)(1+x^{2^1}) = \frac{1 - x^{2^2}}{1-x},$$

as desired.

Inductive Step. Suppose that for $n \geq 2$,

$$(1+x)(1+x^2)(1+x^4)\cdots(1+x^{2^{n-1}}) = \frac{1-x^{2^n}}{1-x}.$$

Multiplying both sides by $1 + x^{2^n}$, we get

$$(1+x)(1+x^2)(1+x^4)\cdots(1+x^{2^n}) = \frac{(1-x^{2^n})(1+x^{2^n})}{1-x} = \frac{1-(x^{2^n})^2}{1-x} = \frac{1-x^{2^{n+1}}}{1-x},$$

as desired.

Induction is complete. □

Part 2.2. Define a sequence $\{a_n\}$ by

$$a_n = \left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{2^2}\right)\cdots\left(1 + \frac{1}{2^{2^n}}\right).$$

Does $\lim_{n\to\infty} a_n$ exist? Determine its value if it does.

Answer. $\lim_{n\to\infty} a_n = 2$.

Solution. By part (a) of this problem with $x = 1/2$,

$$a_n = \frac{1-(1/2)^{2^{n+1}}}{1-(1/2)} = 2\left(1 - \frac{1}{2^{2^{n+1}}}\right).$$

As $n \to \infty$, $2^{n+1} \to \infty$, and therefore, $\frac{1}{2^{2^{n+1}}} \to 0$, leaving $a_n \to 2(1-0) = 2$, as desired. □

Problem 3. True-false problems.

Part 3.1. Let $S = \{\frac{1}{n} | n \in \mathbb{N}\}$. Then $\sup S = 1$.

Solution. This is **true**. Indeed, since $n \geq 1$, $\frac{1}{n} \leq 1$, so it is an upper bound. Finally, $1 = \frac{1}{1} \in S$, so there is no smaller upper bound (and it is in fact the maximum). □

Part 3.2. A function is integrable if and only if its upper integral is less than or equal to its lower integral.

Solution. This is **true**. By definition, a function is integrable if and only if its upper integral is equal to its lower integral. Since the upper integral is always at least the lower integral (as integrals of step functions $s \leq t$), the upper integral is never less than the lower integral and therefore this statement is equivalent to the definition. □

Part 3.3. There are exactly 10 distinct functions $f : \{1, 2\} \to \{1, 2, 3, 4, 5\}$.

Solution. This is **false**. Each of $f(1)$ and $f(2)$ can take on 5 different values, so there are a total of $5 \cdot 5 = 25$ possible functions. □

Part 3.4. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Then

$$\frac{d}{dx}\int_{-x}^{x} f(t)\,dt = f(x) - f(-x).$$

Solution. This is **false**. Let $g(x) = \int_0^x f(t)\,dt$, so

$$\int_{-x}^{x} f(t)\,dt = \int_{-x}^{0} f(t)\,dt + \int_{0}^{x} f(t)\,dt = g(x) - g(-x).$$

Then by the Fundamental Theorem of Calculus, $g'(x) = f(x)$, so

$$\frac{d}{dx}\int_{-x}^{x} f(t)\,dt = g'(x) - (-g'(-x)) = f(x) + f(-x).$$

This also makes intuitive sense: increasing x expands the interval at both ends, so the change in the integral will sum the value of the function at both ends. □

Part 3.5. Let S be the union of the intervals $[0,1]$ and $[2,4]$. Then any continuous function $f : S \to \mathbb{R}$ is bounded.

Solution. This is **true**. Any continuous function on a closed interval is bounded, so $\exists m_1, M_1 \in \mathbb{R}$ such that $m_1 < f(x) < M_1$ on $x \in [0,1]$. Similarly, $\exists m_2, M_2 \in \mathbb{R}$ such that $m_2 < f(x) < M_2$ on $x \in [2,4]$. Therefore, $\min(m_1, m_2) < f(x) < \max(M_1, M_2)$ on $[0,1] \cup [2,4]$, and f is bounded. □

Part 3.6. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $\lim_{x\to 2} f(x) = 3$. Then there exists $\delta > 0$ such that

$$2.9 < f(x) < 3.01 \text{ whenever } 0 < |x-2| < \delta.$$

Solution. This is **true**. By the definition of continuity with $\epsilon = .01$, there exists a δ such that if $0 < |x-2| < \delta$, $2.99 = 3 - .01 < f(x) < 3 + .01 = 3.01$. If $f(x) > 2.99$, then $f(x) > 2.9$, so the uneven inequality $2.9 < f(x) < 3.01$ also holds for all x such that $0 < |x-2| < \delta$. □

Problem 4. Determine the minimum and maximum values of the function $f(x) = x + x(\log x)^2$.

Solution 1. We compute the derivative

$$f'(x) = 1 + (\log x)^2 + x\frac{2\log x}{x} = 1 + 2\log x + (\log x)^2 = (1 + \log x)^2.$$

Since this is nonnegative, f is increasing, and therefore takes on its extremal values at the endpoints of the interval:

$$f\left(\frac{1}{e^2}\right) = \frac{1}{e^2} + \frac{\log(e^{-2})^2}{e^2} = \frac{1}{e^2} + \frac{4}{e^2} = \frac{5}{e^2}$$
$$f(1) = 1 + 1(\log 1)^2 = 1 + 0 = 1.$$

□

Solution 2. We again compute the derivative:

$$f'(x) = 1 + (\log x)^2 + x\frac{2\log x}{x} = 1 + 2\log x + (\log x)^2 = (1 + \log x)^2.$$

Since this exixts for all $x \in [1/e^2, 1]$, the extrema must occur at endpionts or critical points, where $f'(x) = 0$. If $f'(x) = 0$, then $1 + \log x = 0$, or $x = \frac{1}{e}$. We then evaluate f at $\frac{1}{e}$ and the endpoints:

$$f\left(\frac{1}{e^2}\right) = \frac{1}{e^2} + \frac{\log(e^{-2})^2}{e^2} = \frac{1}{e^2} + \frac{4}{e^2} = \frac{5}{e^2}$$
$$f\left(\frac{1}{e}\right) = \frac{1}{e} + \frac{(\log e)^2}{e} = \frac{1}{e} + \frac{1}{e} = \frac{2}{e}$$
$$f(1) = 1 + 1(\log 1)^2 = 1 + 0 = 1.$$

Since $e = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \ldots > 1 + 1 + \frac{1}{2} = \frac{5}{2}$, we have $\frac{5}{e^2} < \frac{2}{e} < \frac{4}{5} < 1$. Therefore, the minimum of f on $[1/e^2, 1]$ is $\frac{5}{e^2}$, and the maximum is 1. $\square$

Problem 5. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Prove that there exists a real number x such that

$$f(x) \sin x - f'(x) \cos x = 0.$$

Solution. Consider the function $g(x) = -f(x) \cos x$ on the interval $[-\pi/2, \pi/2]$. As a product of differentiable functions, $g(x)$ is differentiable with derivative

$$g'(x) = f(x) \sin x - f'(x) \cos x.$$

Since $\cos(-\pi/2) = \cos(\pi/2) = 0$, $g(x) = 0$ on the endpoints of this interval. Therefore, by the Mean Value Theorem, $g'(x) = 0$ somewhere on $(-\frac{\pi}{2}, \frac{\pi}{2})$, and hence, $f(x) \sin x - f'(x) \cos x = 0$, as desired. $\square$

Problem 6.

Part 6.1. Compute the integral

$$\int \sqrt{4 - x^2}\, dx.$$

Answer.

$$\int \sqrt{4 - x^2}\, dx = 2 \sin^{-1}(x/2) + \frac{1}{2} x \sqrt{4 - x^2}.$$

Solution. We start with the substitution $x = 2 \sin \theta$, so $dx = 2 \cos \theta \, d\theta$, and $\sqrt{4 - x^2} = 2\sqrt{1 - \sin^2 \theta} = 2 \cos \theta$. Therefore,

$$\int \sqrt{4 - x^2}\, dx = 4 \int \cos^2 \theta \, d\theta.$$

We then substitute with a double-angle formula: $\cos 2\theta = 2 \cos^2 \theta - 1$, so $\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$ and

$$\int \cos^2 \theta \, d\theta = \int \frac{1 + \cos 2\theta}{2} \, d\theta = \frac{\theta}{2} + \frac{\sin 2\theta}{4} = \frac{1}{2}(\theta + \sin \theta \cos \theta).$$

applying another double-angle formula.[1] Therefore, substituting $\theta = \sin^{-1}(x/2)$ back in, we get the claimed expresion:

$$\int \sqrt{4 - x^2}\, dx = 4 \left(\frac{1}{2} \sin^{-1}(x/2) + \frac{1}{2} \cdot \frac{x}{2} \cdot \sqrt{1 - \frac{x^2}{4}} \right) = 2 \sin^{-1}(x/2) + \frac{1}{2} x \sqrt{4 - x^2}.$$

$\square$

Part 6.2. Compute the integral

$$\int_0^2 x \sqrt{4 - x^2}\, dx.$$

Answer. 8/3.

Solution. This one is much simpler. Substituting $u = x^2$, $du = 2x\, dx$, so

$$\int_0^2 x \sqrt{4 - x^2}\, dx = \int_0^4 \frac{1}{2} \sqrt{4 - u}\, du = \left(-\frac{1}{3} (4 - u)^{3/2} \right) \Bigg|_0^4 = 0 - \left(-\frac{8}{3} \right) = \frac{8}{3}.$$ $\square$

[1] Integration by parts is an alternative approach that does not utilize double-angle formulas.

Problem 7.

Part 7.1. Compute the limit

$$\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}.$$

Answer. This limit is 1.

Solution 1. Since $\sqrt{1+x}-\sqrt{1-x} \to \sqrt{1}-\sqrt{1} = 0$ as $x \to 0$, we can apply L'Hôpital's Rule:

$$\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x} = \lim_{x\to 0} \left(\frac{1}{2\sqrt{1+x}} + \frac{1}{2\sqrt{1-x}}\right) = \left(\frac{1}{2\sqrt{1}} + \frac{1}{2\sqrt{1}}\right) = 1. \qquad \square$$

Solution 2. Some tricks with the conjugate.

$$\frac{\sqrt{1+x}-\sqrt{1-x}}{x} = \frac{(1+x)-(1-x)}{x(\sqrt{1+x}+\sqrt{1-x})} = \frac{2}{\sqrt{1+x}+\sqrt{1-x}}.$$

Therefore, taking a limit,

$$\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x} = \frac{2}{\sqrt{1}+\sqrt{1}} = 1. \qquad \square$$

Part 7.2. Find real numbers A, B, C such that

$$\sqrt{1+x}-\sqrt{1-x} = A + Bx + Cx^2 + o(x^2) \text{ as } x \to 0.$$

Answer. $A = 0, B = 1, C = 0$.

Solution. Letting $f(x) = \sqrt{1+x}-\sqrt{1-x}$, this is just finding the Taylor expansion of f about 0. Since $f(0) = \sqrt{1}-\sqrt{1} = 0$, we must have $A = 0$. With $f'(0) = \lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x} = 1$ from the previous part, $B = 1$ and it remains to calculate $f''(0) = 2C$. This can be done by hand:

$$f''(x) = \frac{d}{dx}\left(\frac{1}{2\sqrt{1+x}} + \frac{1}{2\sqrt{1-x}}\right) = \frac{1}{4(1+x)^{3/2}} - \frac{1}{4(1+x)^{3/2}} \implies f''(0) = \frac{1}{4\cdot 1^{3/2}} - \frac{1}{4\cdot 1^{3/2}} = 0.$$

Or, alternatively, we can observe that $f(-x) = \sqrt{1-x}-\sqrt{1+x} = -f(x)$, so f is odd and therefore, $f''(0) = 0$. Either way, $2C = 0$, so $C = 0$. $\square$

Problem 8. True-false problems, round 2.

Part 8.1. Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Then

$$\int_0^1 xf(x)\,dx = \int_0^1 \left(\int_x^1 f(t)\,dt\right)\,dx.$$

Solution. This is **true**, and is just a consequence of integration by parts. To see this, let $g(x) = \int_1^x f(t)\,dt$. By the Fundamental Theorem of Calculus, $g'(x) = f(x)$, so the left side is just $\int_0^1 xg'(x)\,dx$. We then apply integration by parts:

$$\int_0^1 xg'(x)\,dx = (1\cdot g(1) - 0\cdot g(0)) - \int_0^1 g(x)\,dx.$$

Since $g(1) = \int_1^1 f(t)\,dt = 0$, we conclude that

$$\int_0^1 xf(x)\,dx = -\int_0^1 \left(\int_1^x f(t)\,dt\right)\,dx = \int_0^1 \left(\int_x^1 f(t)\,dt\right)\,dx,$$

as desired. $\square$

Part 8.2. If f is an antiderivative of the rational function $\frac{1}{2x+1}$, then there exists a real number C such that

$$f(x) = \frac{1}{2}\log|2x+1| + C \text{ for all } x \neq -\frac{1}{2}.$$

Solution. This is **false**. It would be true by the Fundamental Theorem of Calculus if our function was defined everywhere on the interval in question. However, across a discontinuity, the constant can change. Concretely, the function

$$f(x) = \begin{cases} \frac{1}{2}\log|2x+1| & x < -\frac{1}{2} \\ \frac{1}{2}\log|2x+1| + 1 & x > -\frac{1}{2} \end{cases}$$

is another antiderivative of $\frac{1}{2x+1}$ that does not take this form. □

Part 8.3. Let $f, g : \mathbb{R} \to \mathbb{R}$ be infinitely differentiable functions. Assume that the fifth Taylor polynomial of f at point 0 is $x^2 + 3x^5$and the fifth Taylor polynomial of g at the point 0 is $x^2 + 2x^5$. Then

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \frac{3}{2}.$$

Solution. This is **false**. In fact, by the definition of the Taylor polynomial, we have

$$\lim_{x \to 0} \frac{f(x)}{x^2} = \lim_{x \to 0} \frac{x^2 + 3x^5}{x^2} = 1$$
$$\lim_{x \to 0} \frac{g(x)}{x^2} = \lim_{x \to 0} \frac{x^2 + 2x^5}{x^2} = 1.$$

Therefore, dividing these two limits,

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \frac{1}{1} = 1 \neq \frac{3}{2}.$$

□

Part 8.4. If $\{a_n\}$ is an increasing sequence with limit L, then $\sup\{a_n | n \in \mathbb{N}\} = L$.

Solution. This is **true**, and was directly covered in class. □

Part 8.5. Any rearrangement of the infinite series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n + (-1)^n)^2} = \frac{1}{1^2} - \frac{1}{5^2} + \frac{1}{5^2} - \frac{1}{9^2} + \frac{1}{9^2} - \cdots$$

converges to 1.

Solution. This is **true**. To prove this is the case, we must show that this series is absolutely convergent, i.e. that this series:

$$\sum_{n=1}^{\infty} \frac{1}{(2n + (-1)^n)^2}$$

converges. Indeed, $2n + (-1)^n \geq 2n - 1 \geq n$, so $\frac{1}{(2n + (-1)^n)^2} \leq \frac{1}{n^2}$. Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so does this series by the comparison test. Therefore, the original alternating series is absolutely convergent, and hence any rearrangement converges to the same value.

Finally, we see that combining consecutive terms, this series telescopes to the value of $1/1^2 = 1$. □

Part 8.6. Let $\{f_n\}$ be a sequence of continuous functions on the interval $[0, 1]$ such that $\lim_{n \to \infty} f_n(x) = 1$ for all $x \in [0, 1]$. Then there exists $M \in \mathbb{R}$ such that f_n is a nonnegative function for all $n > M$.

Solution. This is **false**. This problem captures the difference between pointwise continuity and uniform continuity. The assumption is pointwise continuity, but we claim uniform continuity. One possible counterexample sequence of functions is

$$f_n(x) = \begin{cases} 1 - 2nx & 0 \le x \le 1/n \\ 2nx - 1 & 1/n \le x \le 2/n \\ 1 & 2/n \le x. \end{cases}$$

These functions are piecewise linear and match at the boundaries between pieces, so they are continuous. For every $x \in [0,1]$, if $x > 0$, $\exists N \in \mathbb{N}$ such that if $n \ge N > 2/x$, i.e. $x > 2/n$ and $f_n(x) = 1$ so $\lim_{n\to\infty} f_n(x) = 1$. If $x = 0$, $f_n(x) = 1$ for all n, so this also holds. However, $f_n(1/n) = 1 - 2n(1/n) = -1$ for all n, so none of the f_n are nonnegative functions. □

Problem 9. Consider the power series

$$\sum_{n=2}^{\infty} \frac{2^n}{n \log n} x^n.$$

For which real numbers x does the series converge? For which x does it converge absolutely?

Answer. It converges on $[-1/2, 1/2)$ and converges absolutely on $(-1/2, 1/2)$.

Solution. First consider $x \ge 0$. Applying the root test, consider

$$\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \sqrt[n]{\frac{2^n}{n \log n} x^n} = \lim_{n\to\infty} \frac{2x}{\sqrt[n]{n \log n}}.$$

Since $\sqrt[n]{n} \to 1$ as $n \to \infty$, and $n \le n \log n \le n^2$, taking nth roots and applying the squeeze theorem,

$$1 \le \lim_{n\to\infty} \sqrt[n]{n \log n} \le 1^2 \implies \lim_{n\to\infty} \sqrt[n]{n \log n} = 1.$$

Therefore, $\lim\limits_{n\to\infty} a_n^{1/n} = 2x$. If $x > \frac{1}{2}$, the root test says that the series does not converge, and if $0 < x < \frac{1}{2}$, the series does converge, as desired.

For $x = \frac{1}{2}$, the test is inconclusive. When $x = \frac{1}{2}$, the series is $\sum\limits_{n=2}^{\infty} \frac{1}{n \log n}$. Since the terms are clearly positive and decreasing, we can use the integral test here. The series converges if and only if the following limit does:

$$\lim_{N\to\infty} \int_2^N \frac{1}{x \log x}\, dx = \lim_{N\to\infty} \int_{\log 2}^{\log N} \frac{1}{u}\, du = \lim_{N\to\infty} \log\log N - \log\log 2.$$

Since log is unbounded, this integral diverges, and so does the series.

Finally, we examine $x < 0$. The sum of absolute values of the series is

$$\sum_{n=2}^{\infty} \left| \frac{2^n}{n \log n} x^n \right| = \sum_{n=2}^{\infty} \frac{2^n}{n \log n} |x|^n,$$

which we've already established converges iff $|x| < \frac{1}{2}$, i.e. $-\frac{1}{2} < x < 0$.

If $x < -\frac{1}{2}$, then we claim that the sequence of terms does not converge to zero, which implies that the series diverges. The series is $a_n = \frac{(2x)^n}{n \log n}$, and here $|2x| > 1$, and exponential terms always dominate linear and logarithmic terms in the limit. Therefore, it will not converge to zero, and the series will not converge if $x < -\frac{1}{2}$.

Finally, we examine $n = -\frac{1}{2}$, when the series is $\sum\limits_{n=2}^{\infty} \frac{(-1)^n}{n \log n}$. This is an alternating series whose terms are strictly decreasing to zero, so by Leibniz's Rule, it converges. (However, this series is not absolutely convergent, as we showed before.) □

Problem 10. Let f be a function represented by a power series $\sum a_n x^n$ on the interval $(-1, 1)$. In other words,

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

for all $x \in (-1, 1)$. Assume that $f'''(0) = 4$ and

$$f'(x) = f(x^2) \text{ for all } x \in (-1, 1).$$

Compute the first eight coefficients $a_0, a_1, \ldots, a_7$ of the power series.

Answer. $(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7) = (2, 2, 0, 2/3, 0, 0, 0, 2/21)$.

Solution. First, we are given that $f'''(0) = 3! \cdot a_3 = 4$, so $a_3 = 2/3$.

Since the Taylor series has a positive radius of convergence, we can differentiate and substitute in and compare coefficients on the given functional equation:

$$\begin{aligned} f'(x) &= a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + 6a_6 x^5 + 7a_7 x^6 + o(x^6) \\ f(x^2) &= a_0 + a_1 x^2 + a_2 x^4 + a_3 x^6 + o(x^6). \end{aligned}$$

Therefore, corresponding terms must align. This gives us a system of equations:

$$\begin{aligned} a_0 &= a_1 \\ 2a_2 &= 0 \implies a_2 = 0 \\ 3a_3 &= a_1 \implies a_0 = a_1 = 3a_3 = 3(2/3) = 2 \\ 4a_4 &= 0 \implies a_4 = 0 \\ 5a_5 &= a_2 = 0 \implies a_5 = 0 \\ 6a_6 &= 0 \implies a_6 = 0 \\ 7a_7 &= a_3 \implies a_7 = a_3/7 = 2/21, \end{aligned}$$

as desired. □