Spanning trees, forests and limit shapes

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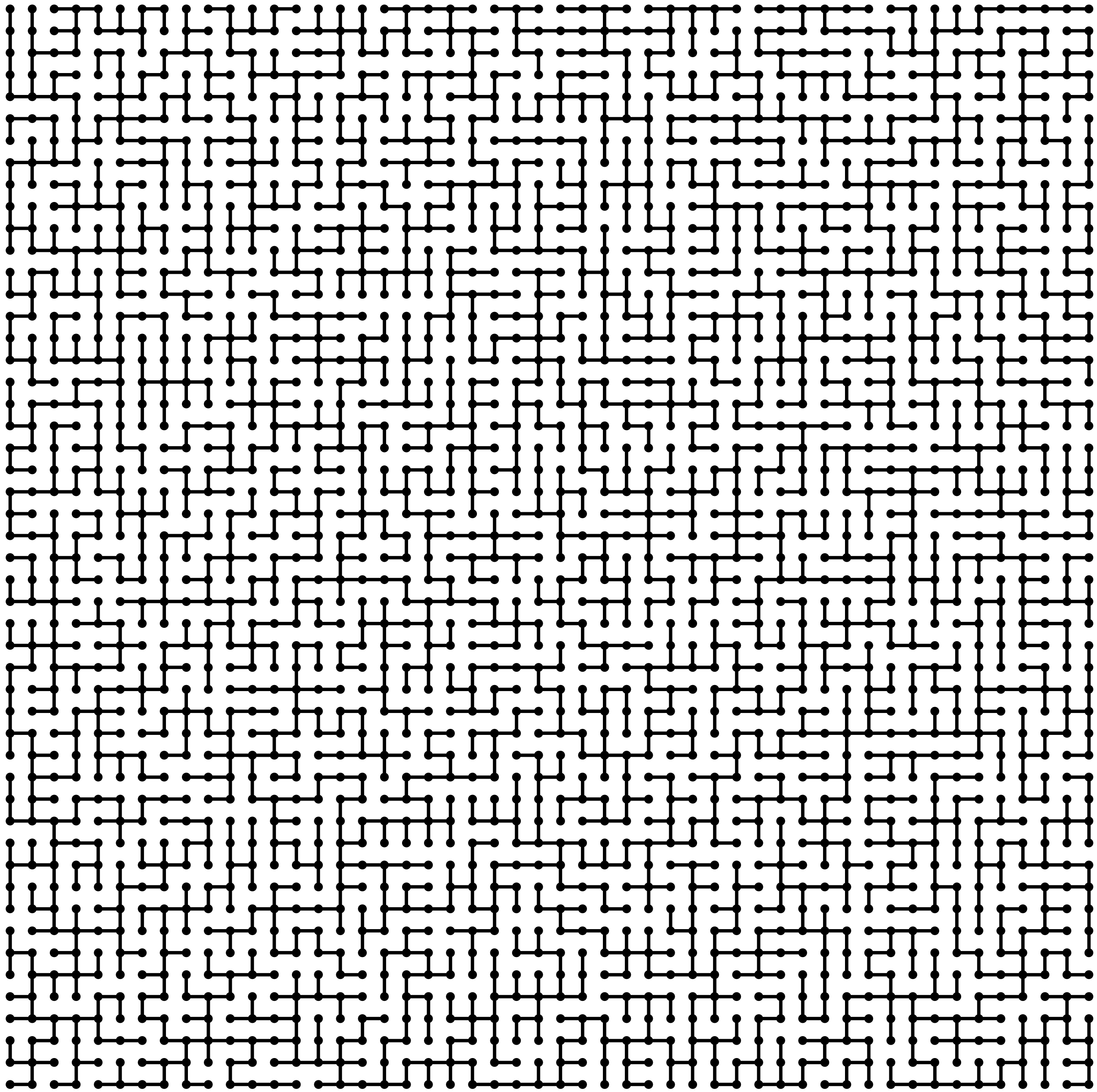
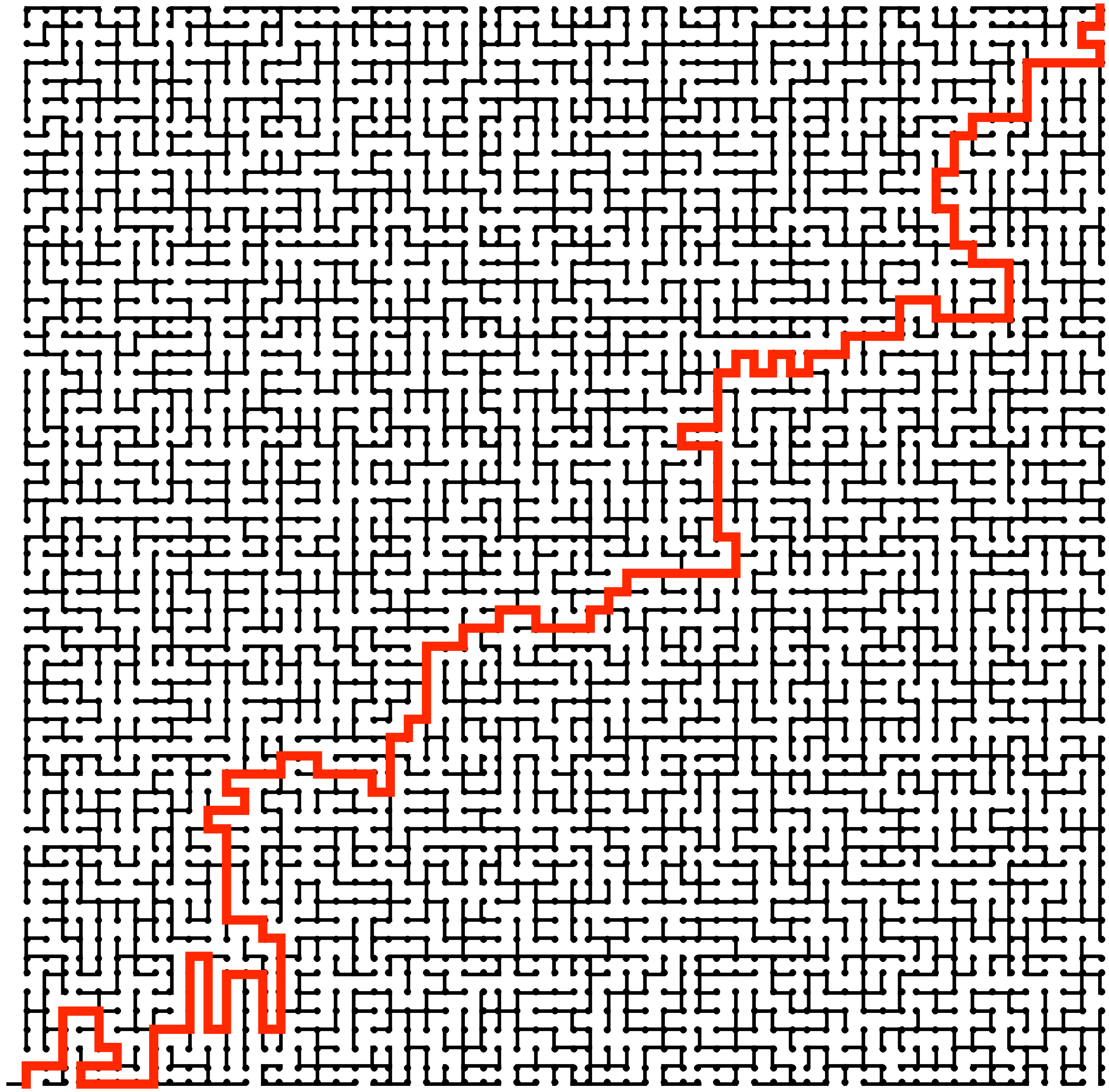


UST on $\mathbb{Z}^2$

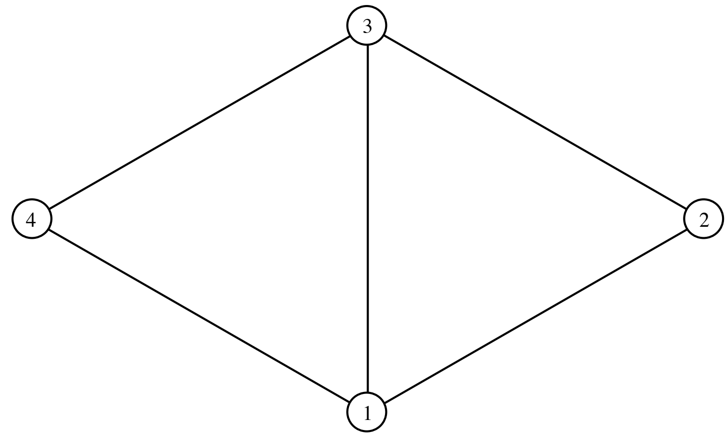


trunk properties (K, Wilson):

$$\text{Prob}(\text{degree} = 2) = \frac{1}{2}$$

$$\text{Prob}(\bullet\!-\!\!-\!\underbrace{\bullet\!-\!\!-\!\bullet\!-\!\!-\!\bullet\!-\!\!-\!\bullet}_{k}) = (\sqrt{2} - 1)^k$$

Given a graph G:



Let $d : \mathbb{R}^V \to \mathbb{R}^E$ be the incidence matrix:

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

Define $\Delta = d^* C d$, where C is a diagonal matrix of conductances.

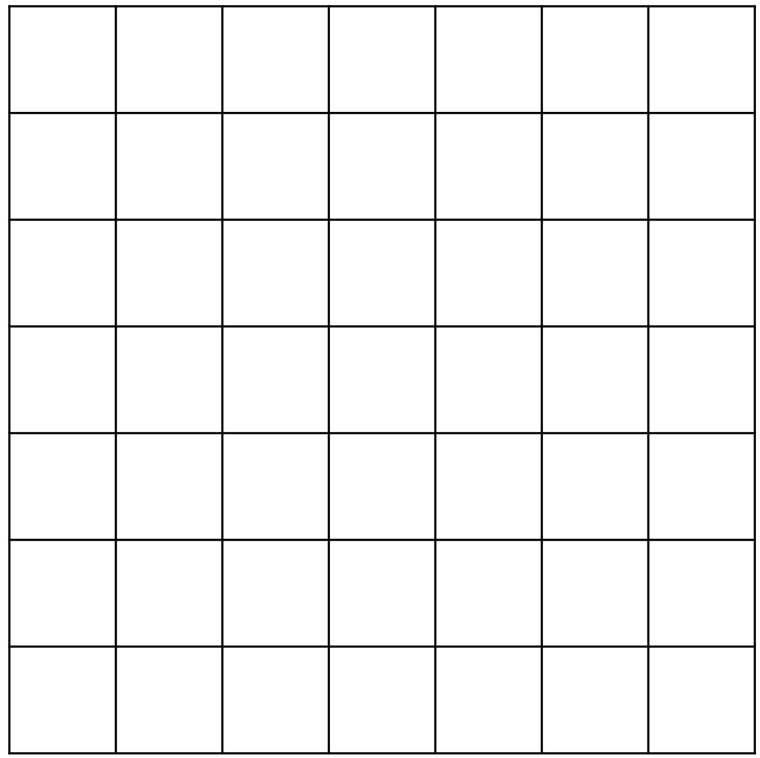
$$\Delta f(v) = \sum_{w \sim v} c_{vw}(f(v) - f(w))$$

Thm (Kirchhoff 1865) $\quad \det \Delta_0 = \sum_{\text{sp. trees}} \prod_{e} c_e$

remove a row and column from Δ

Example $G = \mathbb{Z}^2$

$$\Delta f(v) = 4f(v) - f(v+1) - f(v+i) - f(v-1) - f(v-i)$$



Δ is a "convolution" operator; its Fourier transform is multiplication by $P(z,w)$:

$$P(z,w) = 4 - z - \frac{1}{z} - w - \frac{1}{w}.$$

Q. What do the roots of $P(z,w) = \hat{\Delta}$ tell us about Δ?

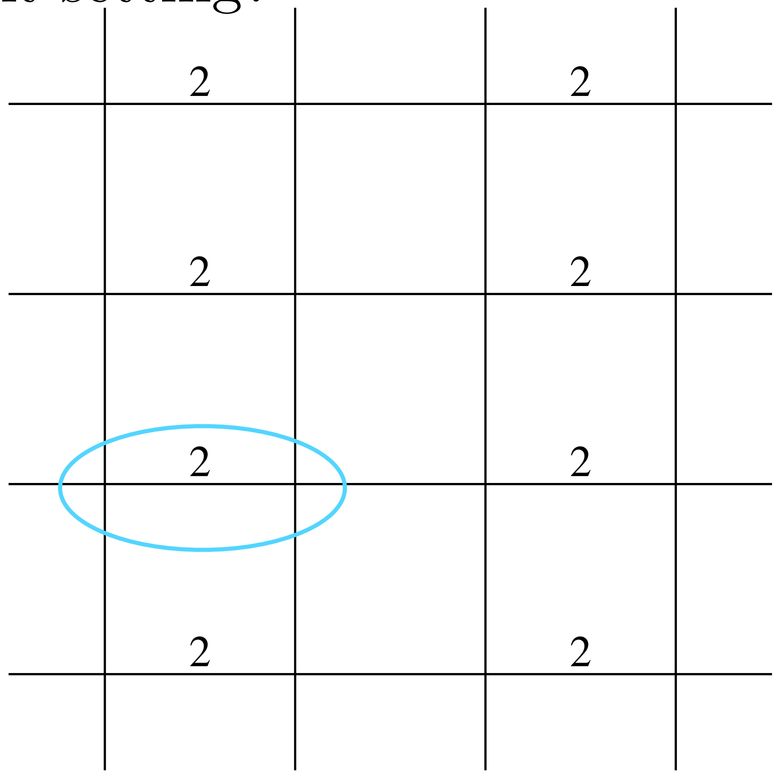
The Green's function (potential kernel)

$$G(x,y) = -\frac{1}{4\pi^2}\int_{\mathbb{T}^2} \frac{z^x w^y - 1}{4 - z - 1/z - w - 1/w}\, dz\, dw$$

For large x, y, the only relevant part of $P = 0$ is near $(1,1)$.

What about a slightly different setting?

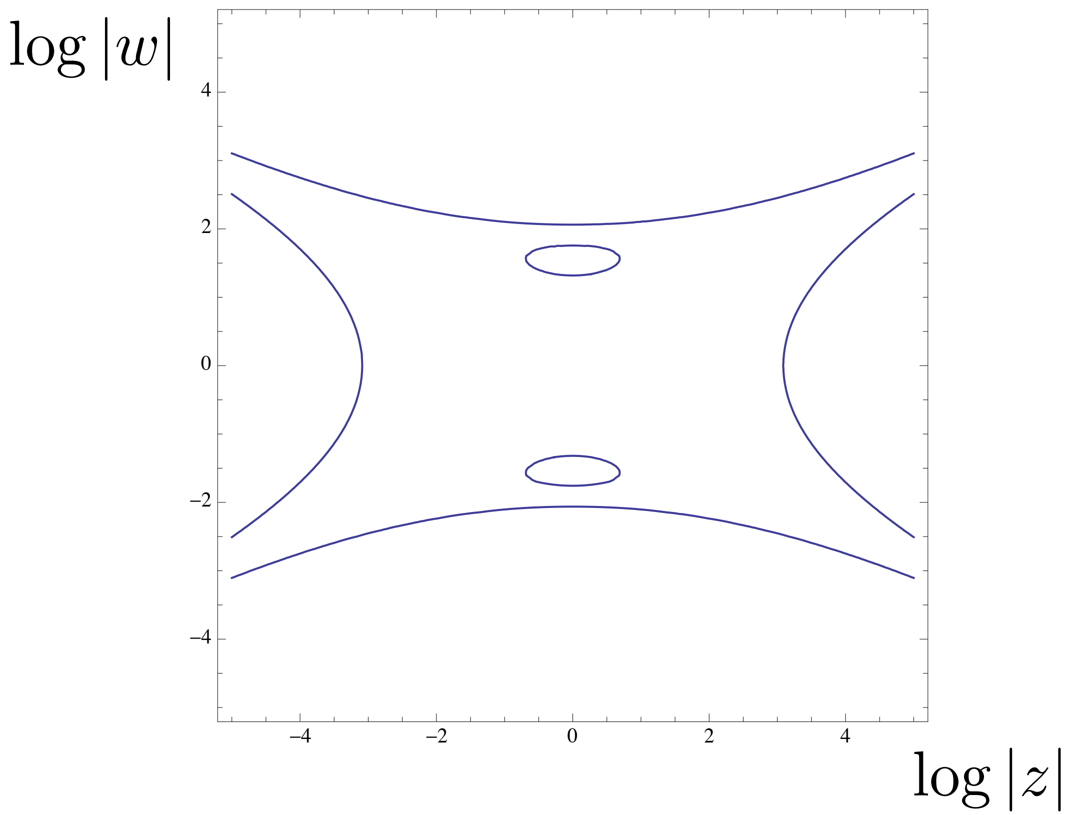
Periodic conductances:



$$\hat{\Delta} = \begin{pmatrix} 5 - w - 1/w & -2 - 1/z \\ -2 - z & 5 - w - 1/w \end{pmatrix}$$

$$P(z, w) = \det \hat{\Delta} = w^2 + \frac{1}{w^2} - 10w - \frac{10}{w} - 2z - \frac{2}{z} + 22$$

$P = 0$ has topology away from $(z, w) = (1, 1)$:



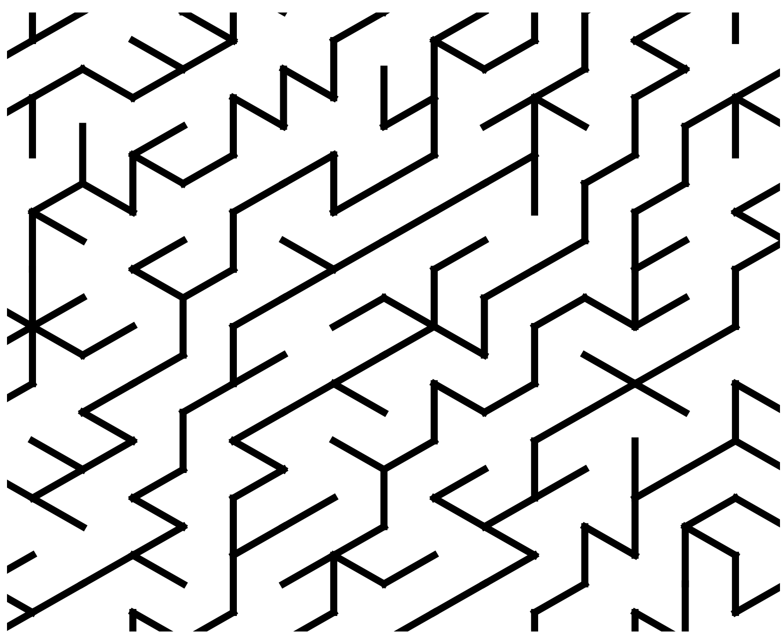
Q. What properties of spanning trees involve other points of $P = 0$?

Q. Combinatorially, what is the meaning of the coefficients of P?

A. The UST is one of a two-parameter family of probability measures indexed by points (z, w) on $P(z, w) = 0$.

$$\text{UST} \leftrightarrow (1, 1)$$

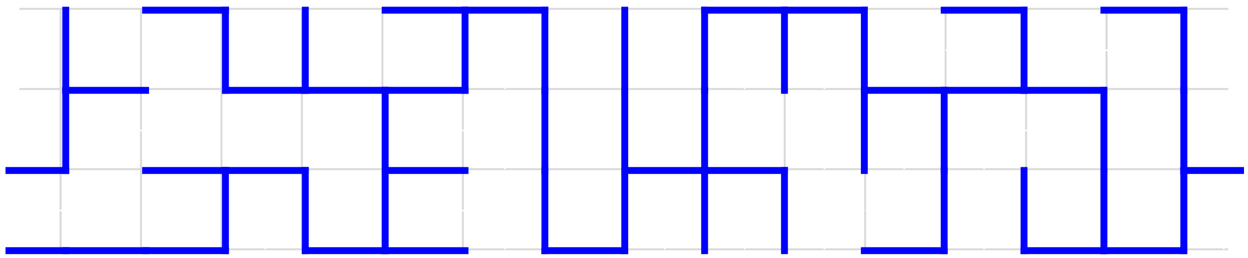
Other points correspond to measures on "essential spanning forests"



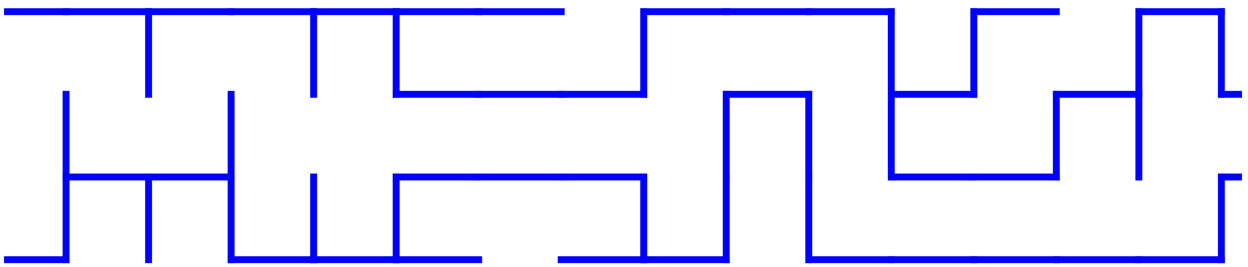
sample configuration from another measure (triangular lattice)

On a strip graph,

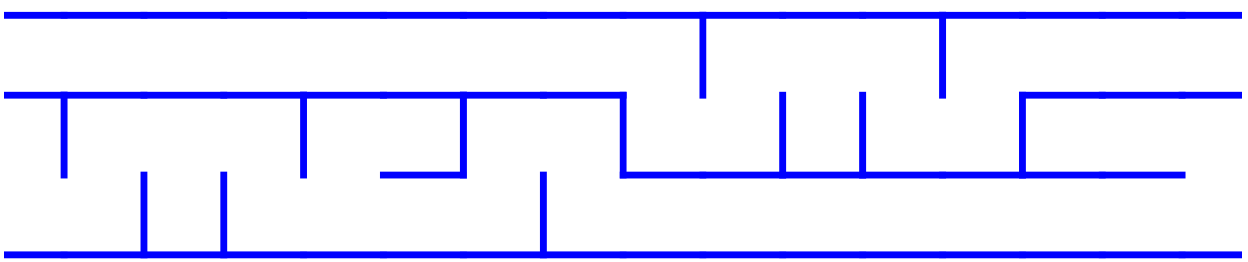
μ_1



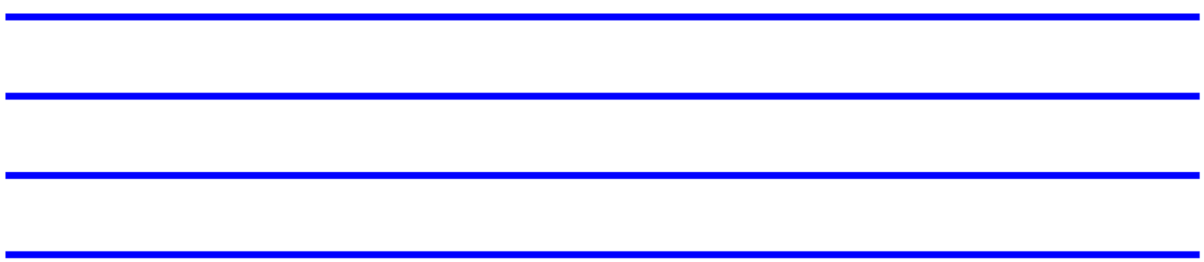
μ_2

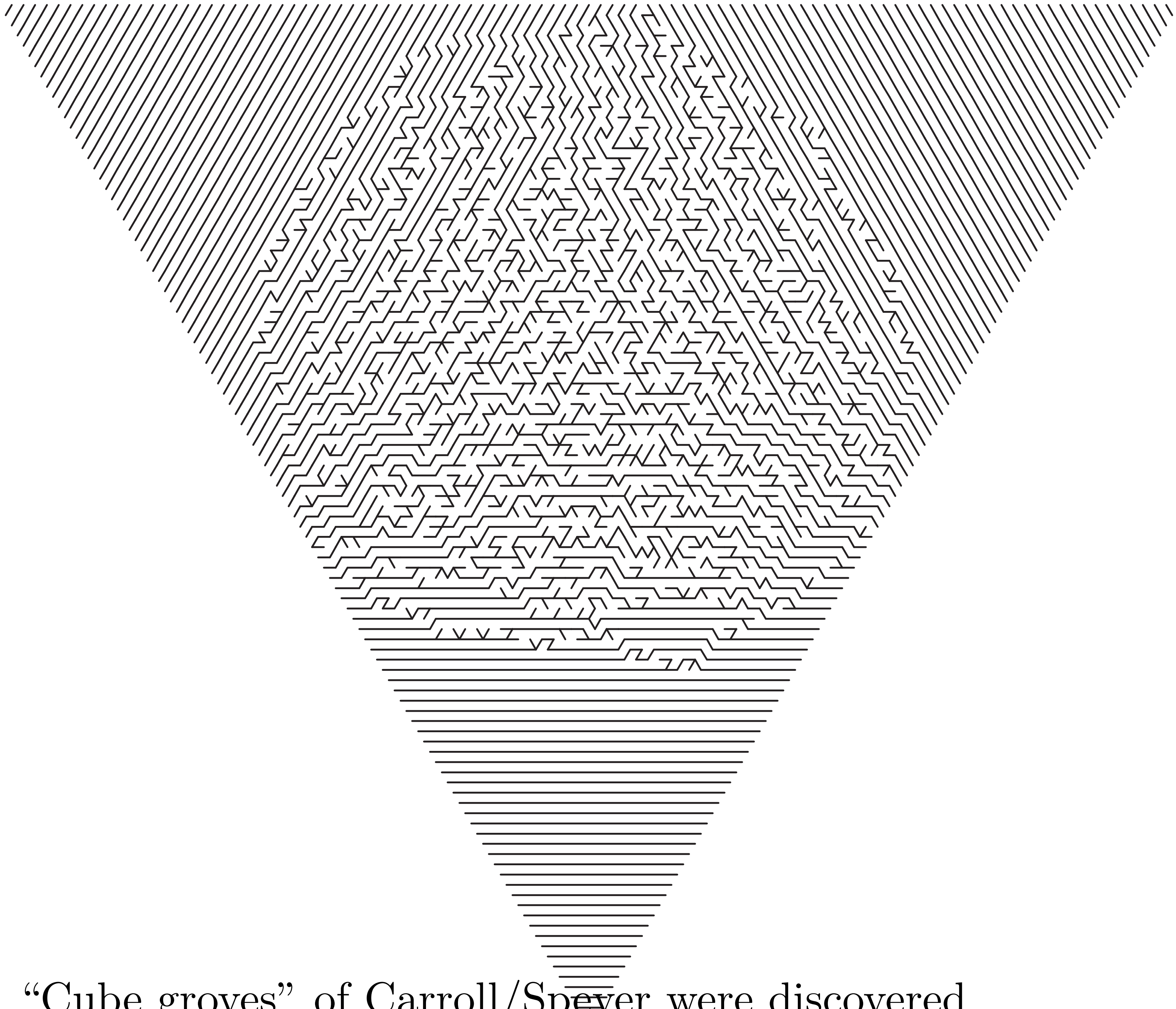


μ_3



μ_4



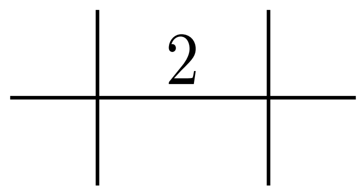


“Cube groves” of Carroll/Speyer were discovered in the study the *cube recurrence.* (2004)

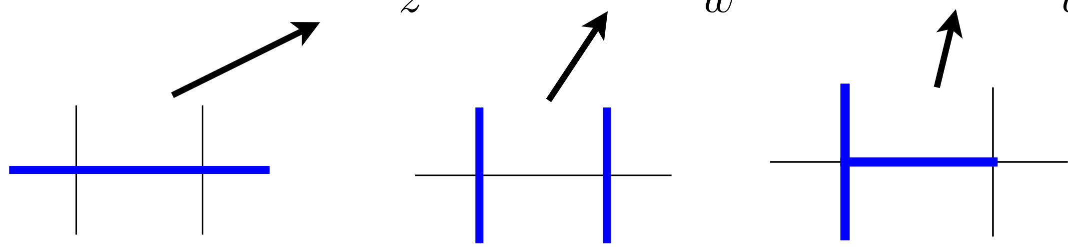
$P(z,w)$ counts *cycle-rooted spanning forests (CRSFs)* on the torus
(subgraphs in which each component is a tree plus one edge)

Thm: $$P(z,w) = \sum_{\text{CRSFs } C} \left(\prod_e c_e\right)(2 - z^i w^j - z^{-i}w^{-j})^k,$$

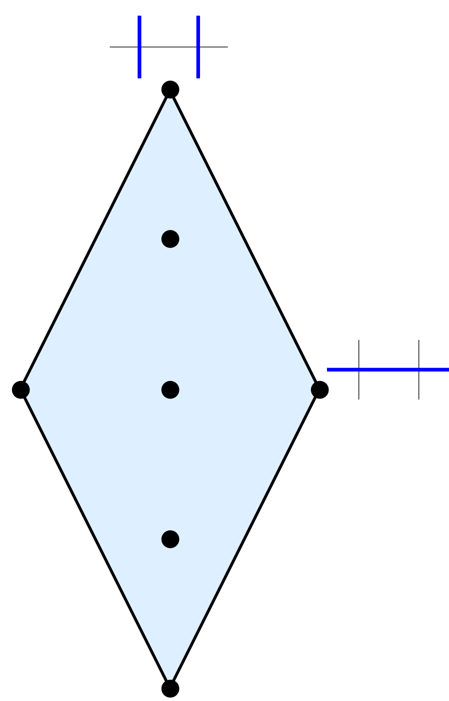
where C has k cycles of homology class (i,j).

e.g. 

$$P(z,w) = w^2 + \frac{1}{w^2} - 10w - \frac{10}{w} - 2z - \frac{2}{z} + 22$$
$$= 2(2 - z - \frac{1}{z}) + (2 - w - \frac{1}{w})^2 + 6(2 - w - \frac{1}{w})$$

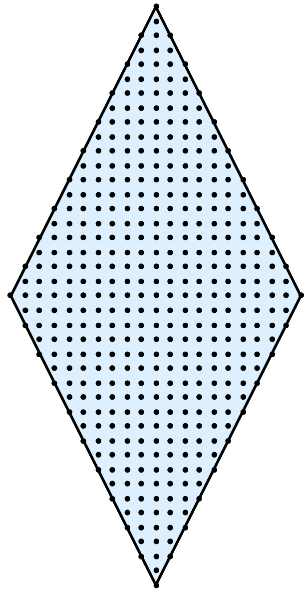


$(+\ldots)$

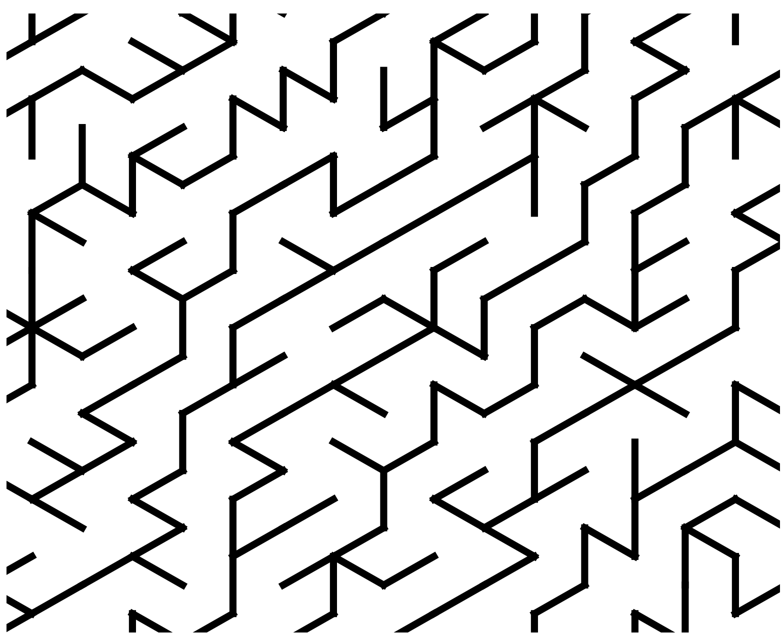
N_1 : 

The set of homology classes forms a convex polygon N:
the Newton polygon of P, symmetric around $(0,0)$

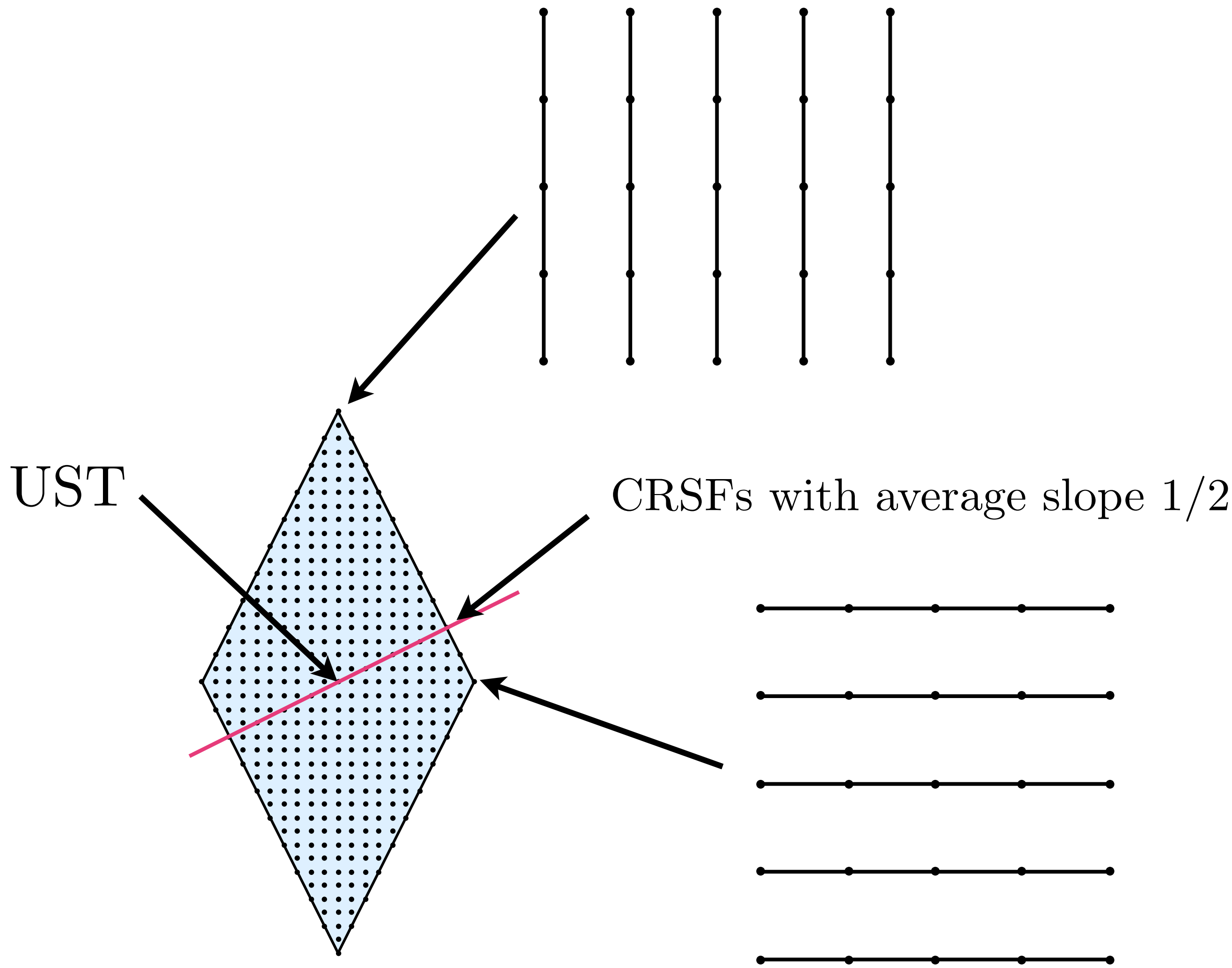
If we enlarge the fundamental domain (take a cover of the torus)

N_{10} :  for the 10×10 cover $\quad (N_{10} = 10N_1)$

Real points (s,t) in N_1 parametrize measures $\mu_{s,t}$ on planar configurations with fixed average slope and density of crossings:



a random sample from $\mu_{.3,.5}$.


UST
CRSFs with average slope 1/2

Thm: The measures $\mu_{s,t}$ are *determinantal**(for edges).

The kernel is given by

$$(K_{s,t})_{e,f} = \frac{1}{4\pi^2} \iint_{|z|=e^X,|w|=e^Y} \frac{K(z,w)_{[e],[f]} z^{x_1-x_2} w^{y_1-y_2}}{P(z,w)} \frac{dz}{iz} \frac{dw}{iw}$$

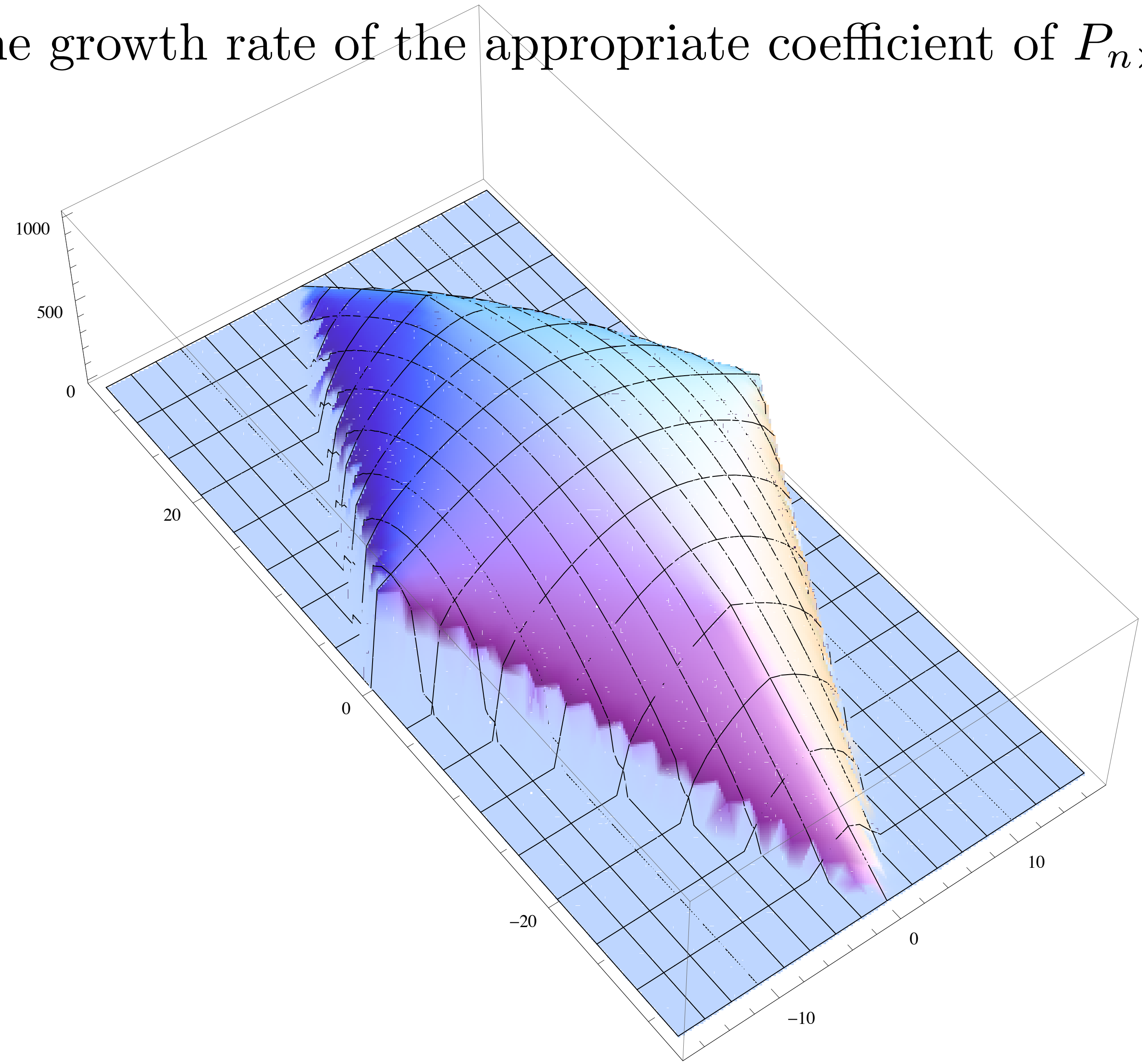
Note: The only dependence on s,t is in contour of integration.

*There is a matrix K such that

$$Pr(e_1,\ldots,e_k \in T) = \det[K(e_i,e_j)_{i,j=1,\ldots,k}]$$

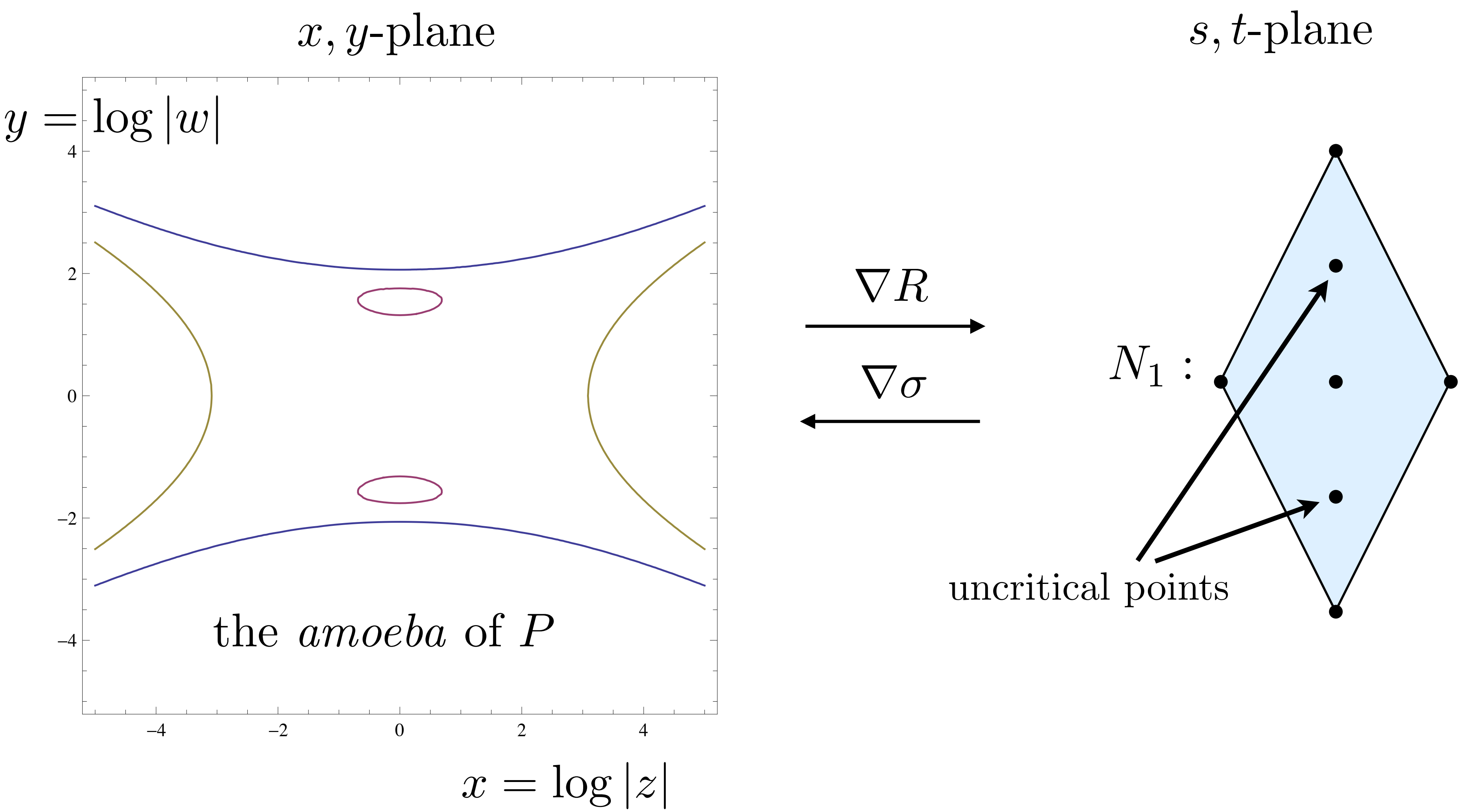
The free energy (growth rate) of $\mu_{s,t}$

is the growth rate of the appropriate coefficient of $P_{n\times n}(z,w)$:



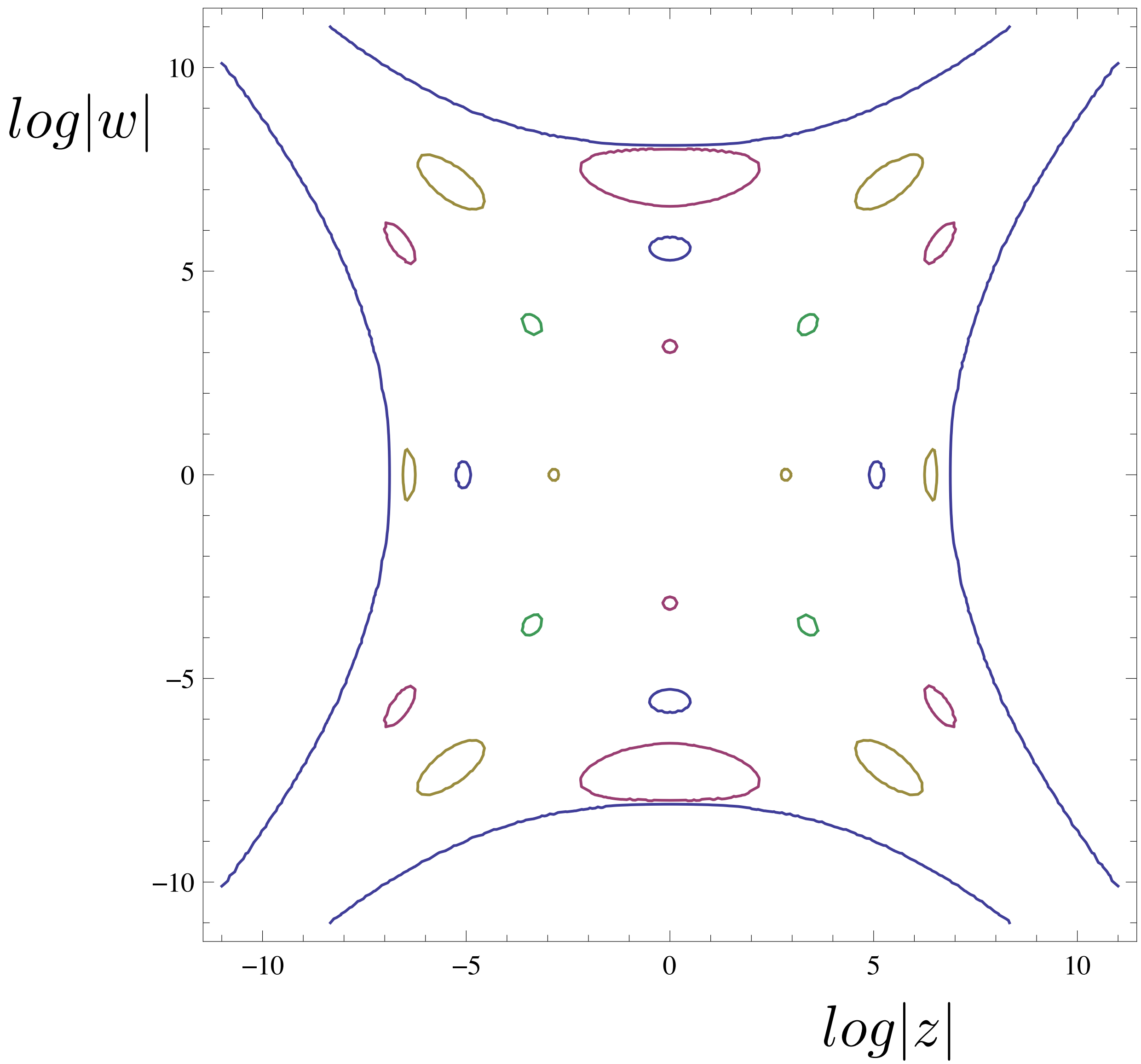
$$\sigma(s,t) = \lim_{n\to\infty} \frac{1}{n^2} \log([z^{sn}w^{tn}]P_{n\times n}(z,w))$$

Legendre duality $(s,t) \leftrightarrow (x,y)$

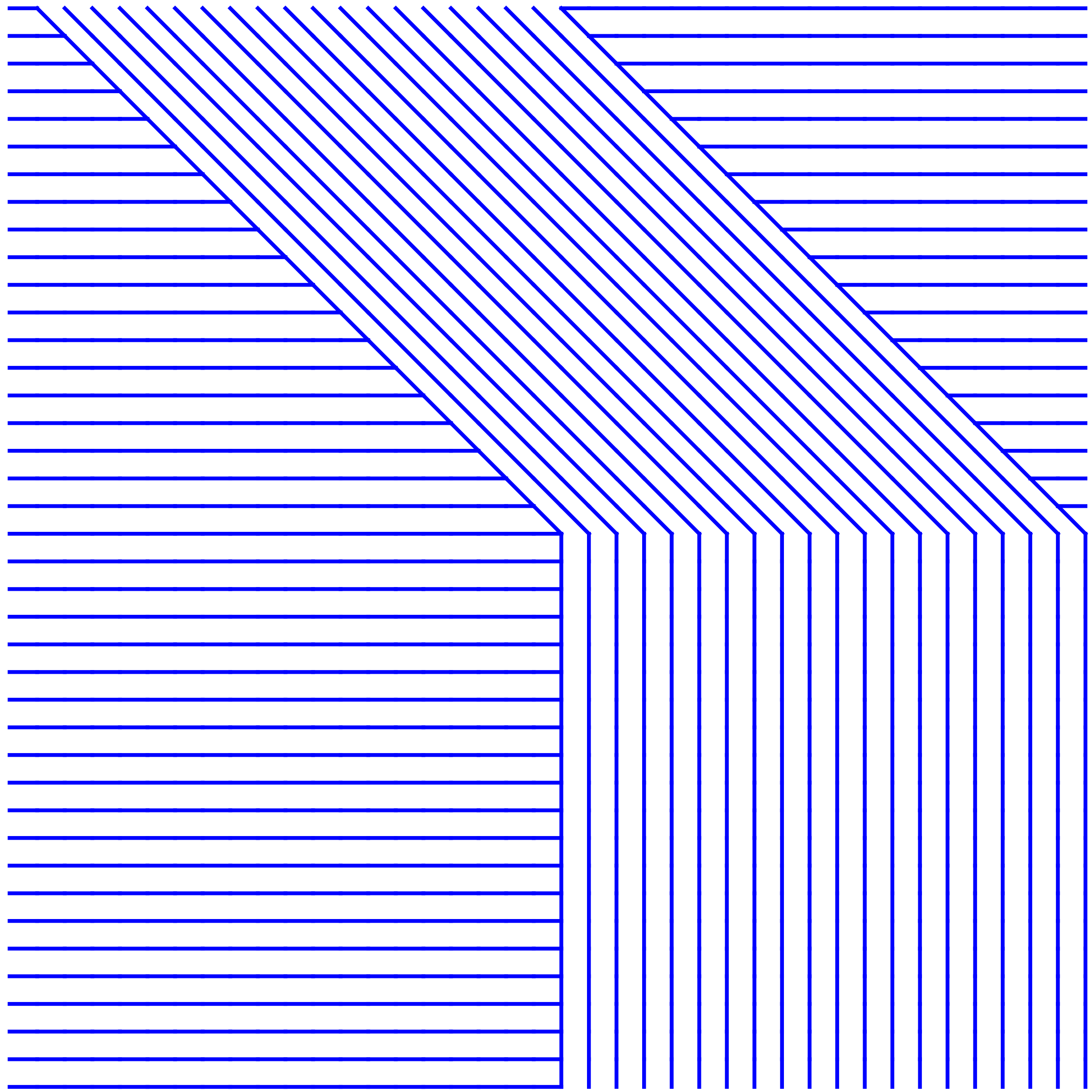


For s, t an uncritical point, $\mu_{s,t}$ has exponential decay of correlations!

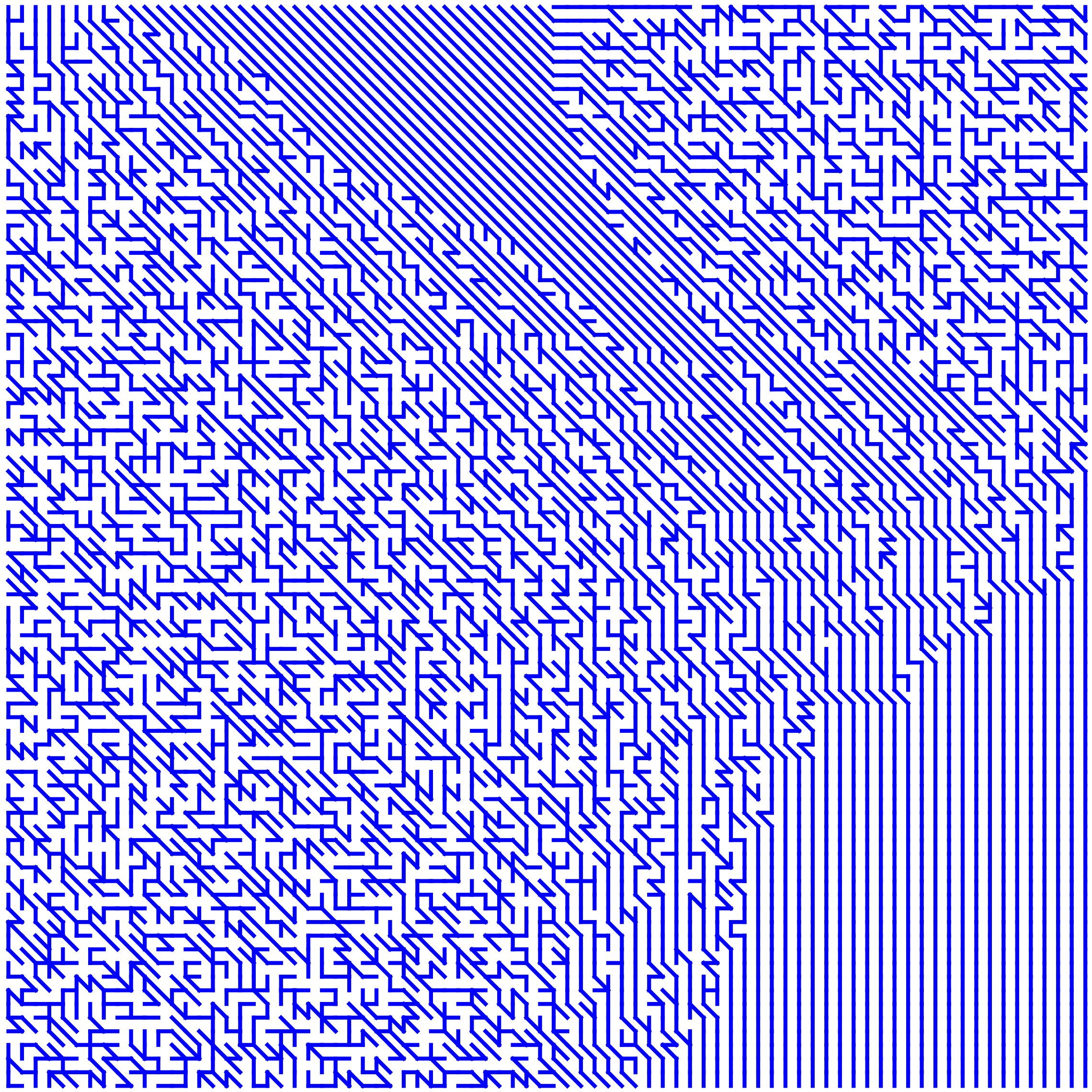
If the graph G has larger fundamental domain, the phase space is richer:



Boundary connections and limit shapes

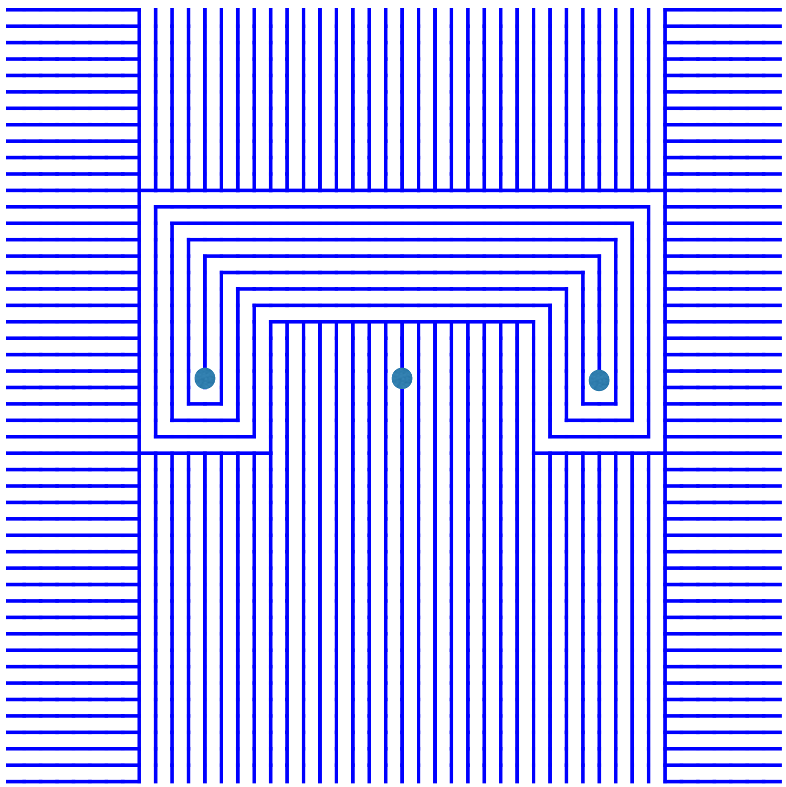


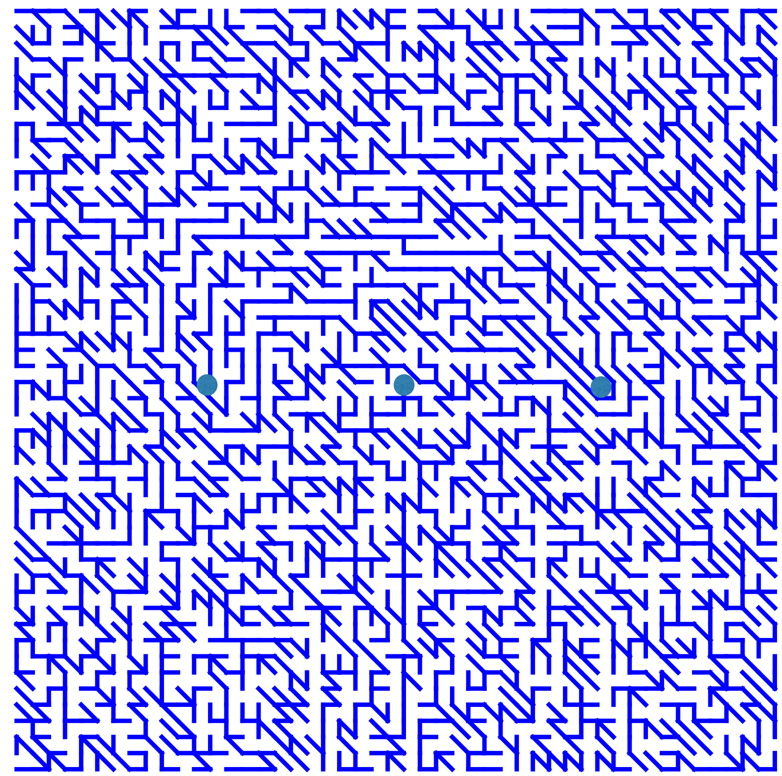
Given a region with a spanning forest connecting certain boundary vertices...



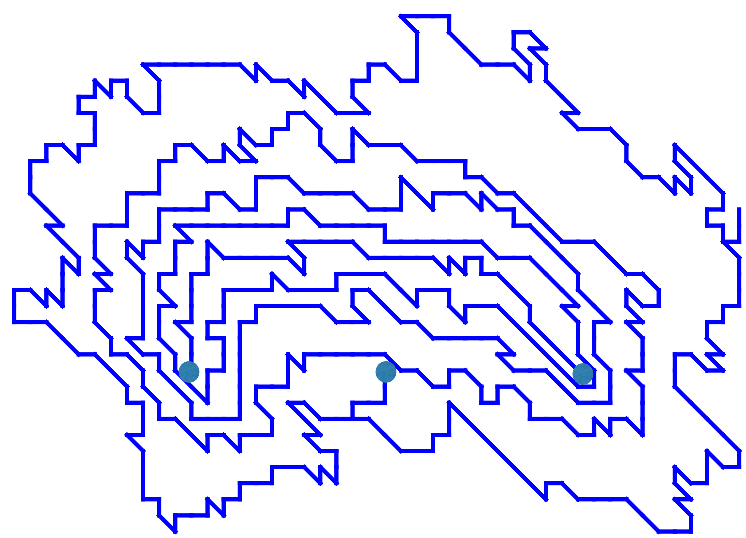
Find a *uniform* spanning forest with the same boundary connections...

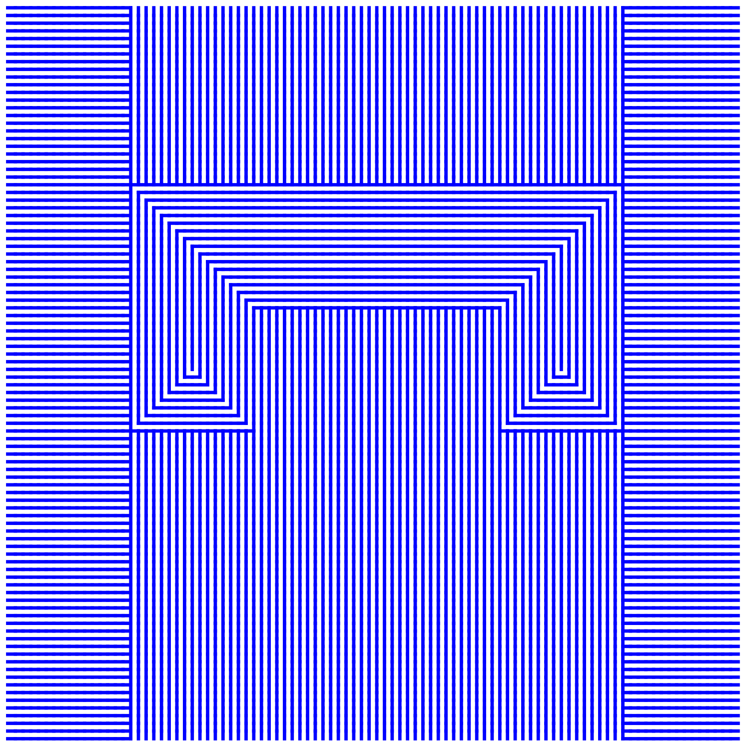
More generally, start with a CRSF on a multiply connected domain

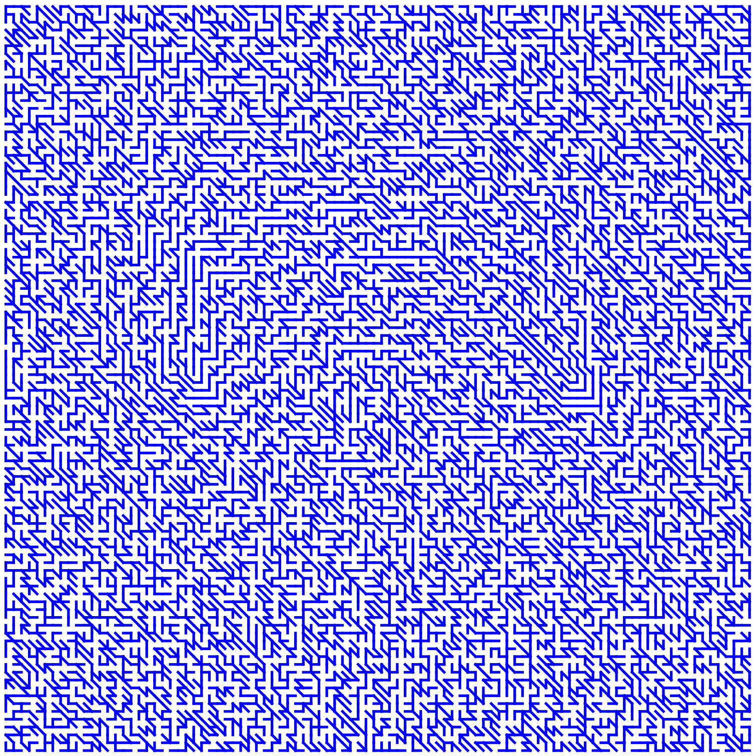


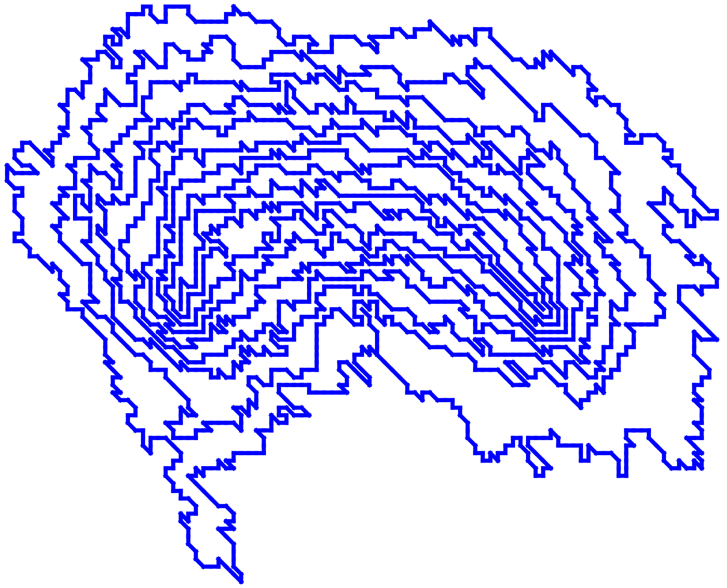


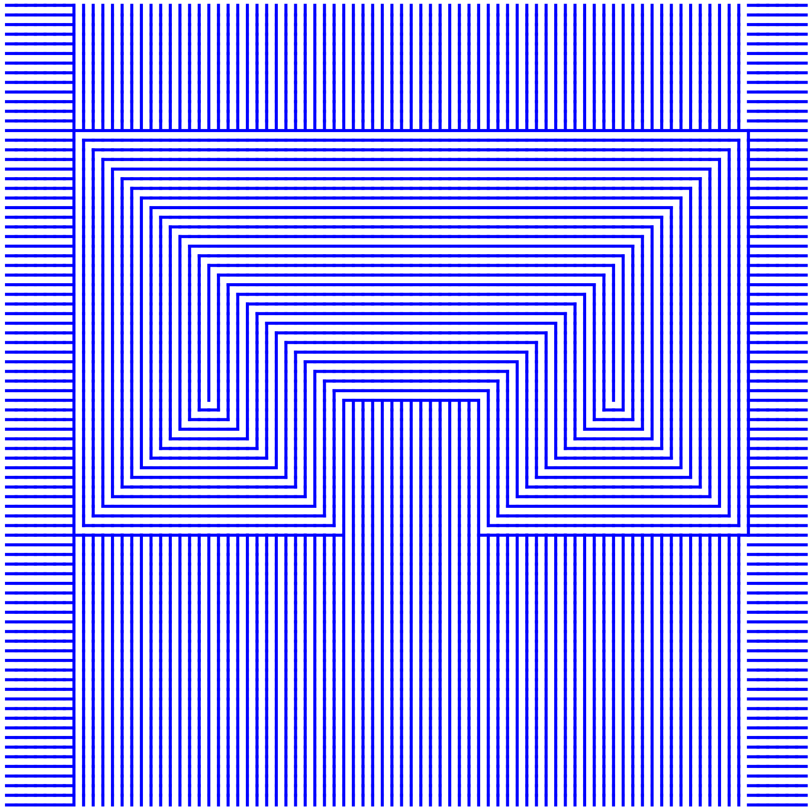
(possibly having certain boundary connections)...and find a uniform sample with same homotopy type and connections.

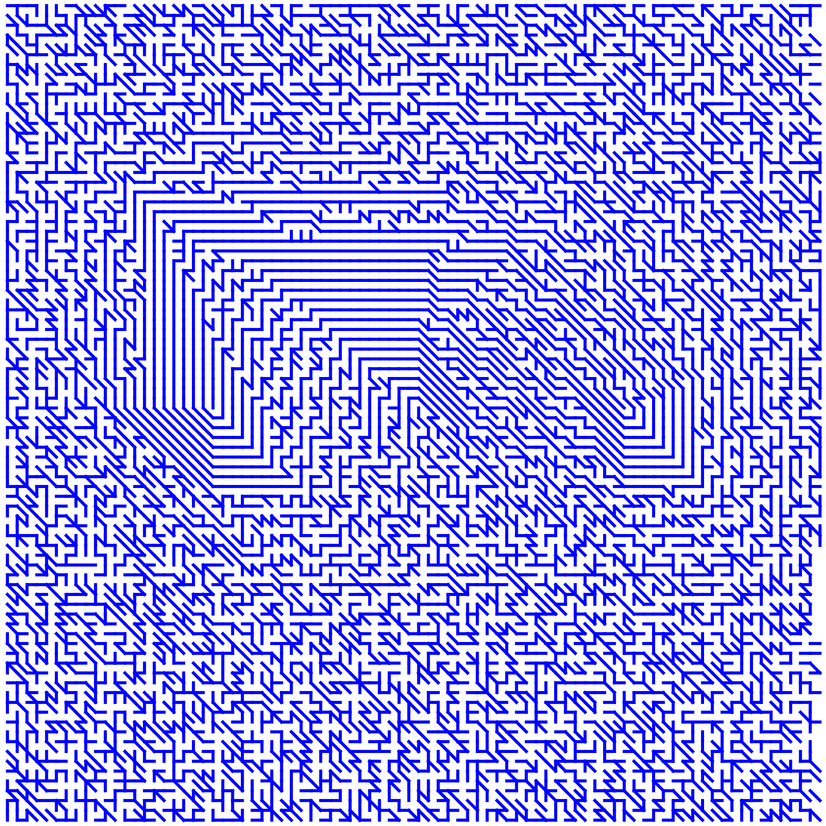


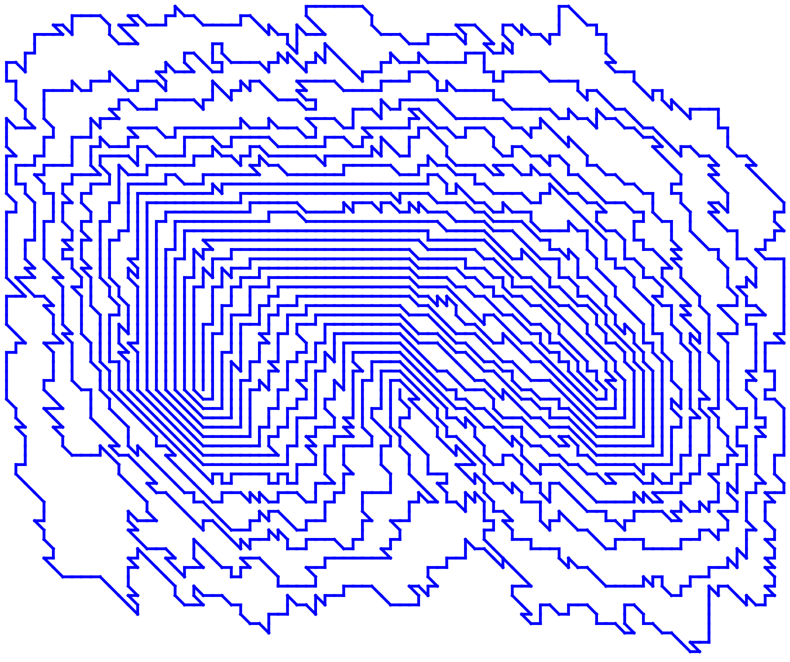












Limit shapes

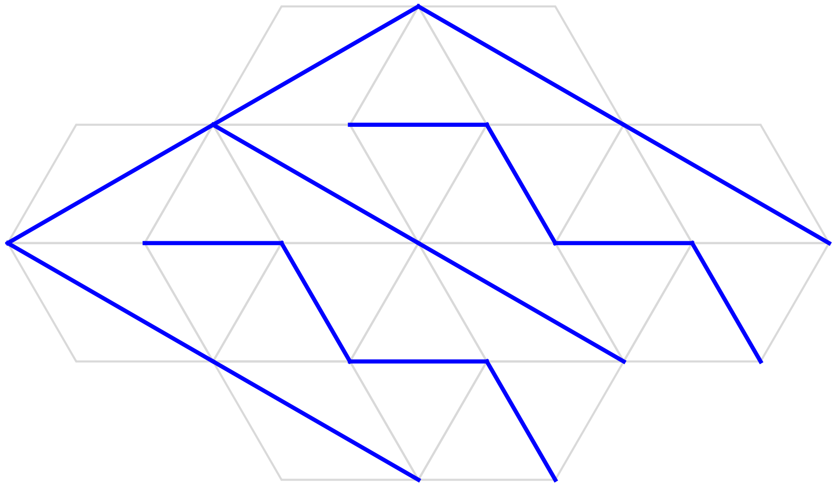
"measured foliation"

Given a domain $U \subset \mathbb{R}^2$ and "unsigned 1-form" $|dy|$ satisfying a certain Lipschitz condition: $|dy(u)|u \in N$ for $u \in S^1$ the limit of the CRSF process $|dy_\epsilon|$ on $U \cap \epsilon\mathbb{Z}^2$ with local slope approximating $|dy|$ exists, and is the unique unsigned 1-form maximizing

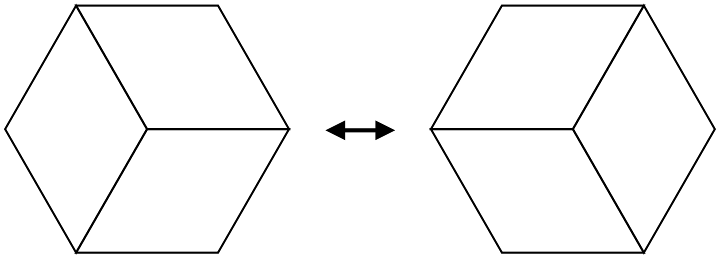
$$\iint_U \sigma(|dy|)\, dA.$$

How to sample CRSFs with given topology?

...MCMC



add and remove "cubes"



whose faces are decorated with spanning tree edges (or dual edges).