Order - disorder operators in planar and almost planar graphs (2)

Hugo Duminil-Copin, *I.H.É.S.*

The main statement

Consider the Ising's Hamiltonian (with free boundary conditions) on a finite subgraph G of the square lattice $\mathbb{Z}^2$ with coupling constants $J_{xy} \geq 0$,

$$H_G(\sigma) \stackrel{\text{def}}{=} -\sum_{x,y \in G} J_{xy} \sigma_x \sigma_y$$

and the associated measure at inverse-temperature β defined for any f,

$$\langle f \rangle_{G,\beta} = \frac{\displaystyle\sum_{\sigma \in \{\pm 1\}^G} f(\sigma) \exp[-\beta H_G(\sigma)]}{\displaystyle\sum_{\sigma \in \{\pm 1\}^G} \exp[-\beta H_G(\sigma)]}.$$

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Theorem (Aizenman, Duminil-Copin, Tassion, Warzel (2016))

For $x_1, \dots, x_{2n}$ found in this order on the boundary of $\mathbb{H}$,

$$\langle \sigma_{x_1}\cdots\sigma_{x_{2n}}\rangle_{\mathbb{H},\beta_c} \sim \mathrm{Pfaff}\Big[\big(\langle\sigma_{x_i}\sigma_{x_j}\rangle_{\mathbb{H},\beta_c}\big)_{1\leq i<j\leq 2n}\Big]$$

as $\min|x_i - x_j|$ tends to infinity.

Part I. Planar Case

Rewriting correlations functions in terms of random currents

Rewrite the model in terms of integer-valued functions (called **currents**) $\mathbf{m} = (\mathbf{m}_{xy} : x, y \in G)$.

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$$\exp[\beta J_{xy}\sigma_x\sigma_y] = \sum_{\mathbf{m}_{xy}=0}^{\infty} \frac{(\beta J_{xy}\sigma_x\sigma_y)^{\mathbf{m}_{xy}}}{\mathbf{m}_{xy}!}$$

allows us to write for $\sigma_A = \prod_{x\in A}\sigma_x$,

$$\sum_{\sigma\in\{\pm1\}^G} \sigma_A \exp[-\beta \mathbf{H}_G(\sigma)] \overset{\text{def}}{=} \sum_{\sigma\in\{\pm1\}^G} \sigma_A \prod_{x,y\in G} \exp[\beta J_{xy}\sigma_x\sigma_y]$$

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$$\sum_{\sigma\in\{\pm1\}^G} \sigma_A \exp[-\beta \mathbf{H}_G(\sigma)] \overset{switch\_sums}{=} \sum_{\mathbf{m}} w(\mathbf{m}) \sum_{\sigma\in\{\pm1\}^G} \prod_{x\in G} \sigma_x^{\mathbb{I}[x\in A]+\Delta(\mathbf{m},x)}$$

where $w(\mathbf{m}) \overset{\text{def}}{=} \prod_{x\sim y} \frac{\beta^{\mathbf{m}_{xy}}}{\mathbf{m}_{xy}!}$ and $\Delta(\mathbf{m},x) \overset{\text{def}}{=} \sum_y \mathbf{m}_{xy}$.

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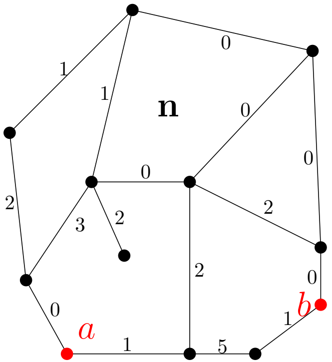
$$\sum_{\sigma\in\{\pm 1\}^G} \sigma_A \exp[-\beta \mathbf{H}_G(\sigma)] \quad = \quad 2^{|G|}\sum_{\partial\mathbf{m}=A} w(\mathbf{m}),$$

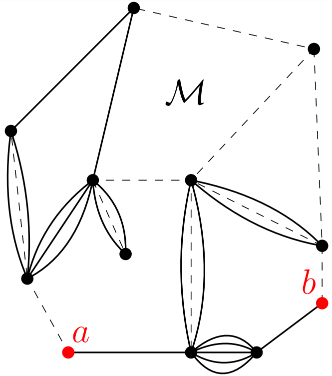
where $w(\mathbf{m}) \stackrel{\text{def}}{=} \prod_{x\sim y} \frac{\beta^{\mathbf{m}_{xy}}}{\mathbf{m}_{xy}!}$ and sources $\partial\mathbf{m} \stackrel{\text{def}}{=} \{x \in G, \Delta(\mathbf{m}, x) \text{ odd}\}$.

based on the fact that for each fixed $x \in G$, the map flipping σ_x is an **involution** on spin configurations.

An interpretation of currents in terms of loops

Identify $\mathbf{m} = (\mathbf{m}_{xy} : x, y \in G)$ with a (multi-)graph $\mathcal{M}$ with $\mathbf{m}_{xy}$ edges between x and y.





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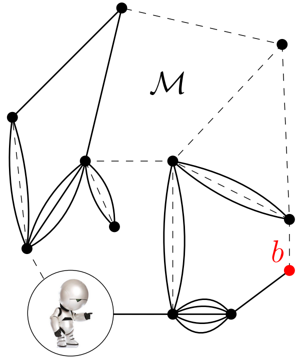

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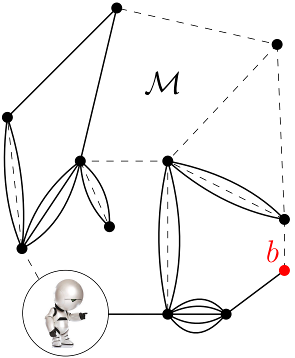

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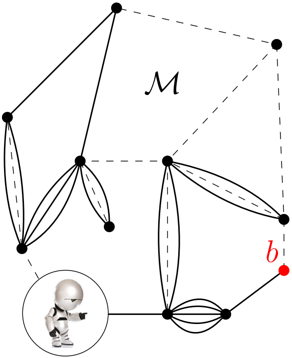

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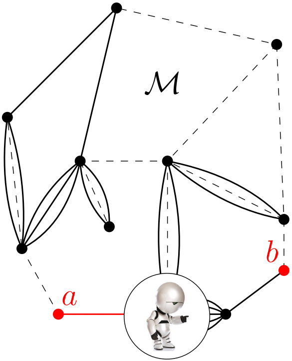

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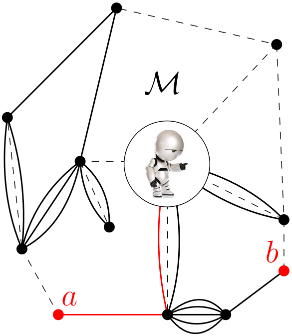

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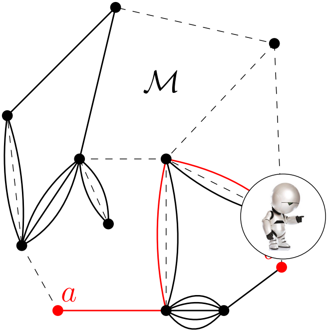

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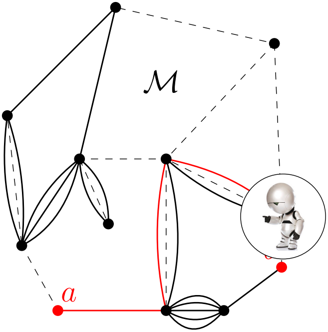

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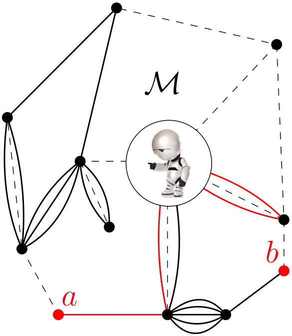

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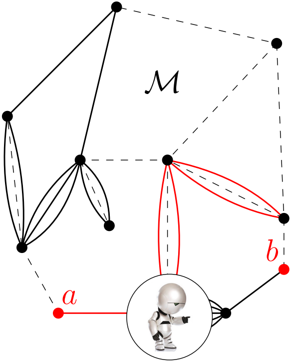

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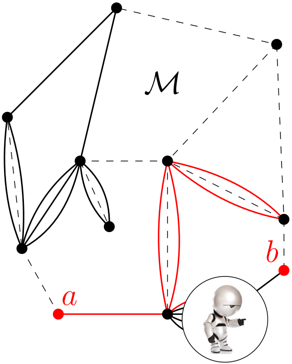

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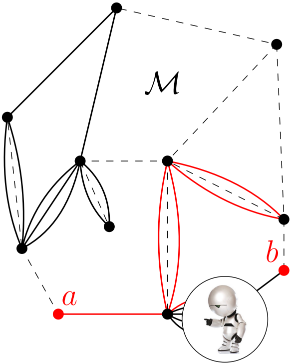

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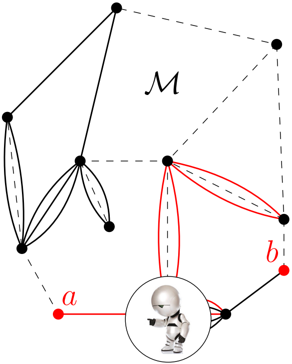

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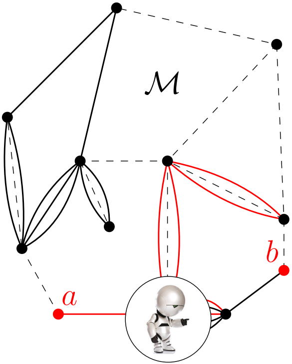

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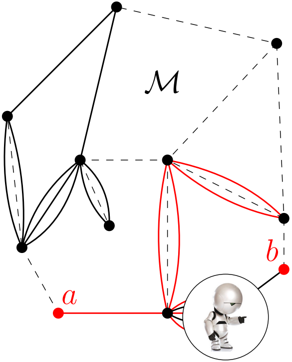

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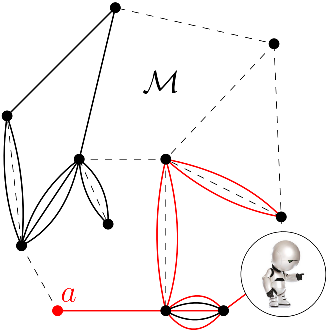

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Lemma (Switching lemma)

For $A \subset G$ and $x, y \in G$

$$\sum_{\substack{\partial \mathbf{m}_1 = A \\ \partial \mathbf{m}_2 = \{x,y\}}} w(\mathbf{m}_1) w(\mathbf{m}_2) = \sum_{\substack{\partial \mathbf{m}_1 = A \Delta \{x,y\} \\ \partial \mathbf{m}_2 = \emptyset}} w(\mathbf{m}_1) w(\mathbf{m}_2) \mathbb{I}[x \overset{\mathcal{M}_1 \cup \mathcal{M}_2}{\longleftrightarrow} y].$$

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$$\sum_{\substack{\partial \mathbf{m}_1 = B \\ \partial \mathbf{m}_2 = A}} F(\mathbf{m}_1, \mathbf{m}_2)w(\mathbf{m}_1)w(\mathbf{m}_2) = \sum_{\partial \mathbf{m} = A\Delta B} w(\mathbf{m}) \sum_{\substack{\partial \mathbf{n} = B \\ \mathbf{n} \le \mathbf{m}}} F(\mathbf{n}, \mathbf{m} - \mathbf{n}) \binom{\mathbf{m}}{\mathbf{n}}.$$

Simply make the change of variables $\mathbf{m} = \mathbf{m}_1 + \mathbf{m}_2$ and $\mathbf{n} = \mathbf{m}_1$, and observe that $w(\mathbf{m}_1)w(\mathbf{m}_2) = w(\mathbf{m})\binom{\mathbf{m}}{\mathbf{n}}$ where $\binom{\mathbf{m}}{\mathbf{n}} := \prod_{x,y} \binom{\mathbf{m}_{xy}}{\mathbf{n}_{xy}}$

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Applications of the switching principle

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$$\overset{\text{switching}}{=} \frac{\sum_{\partial\mathbf{m}_1=\partial\mathbf{m}_2=\emptyset} w(\mathbf{m}_1)w(\mathbf{m}_2)\mathbb{I}[x \overset{\mathcal{M}_1\cup\mathcal{M}_2}{\longleftrightarrow} y]}{\sum_{\partial\mathbf{m}_1=\partial\mathbf{m}_2=\emptyset} w(\mathbf{m}_1)w(\mathbf{m}_2)}$$

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so that it is sufficient to prove that

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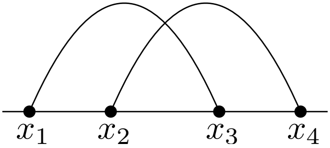
For a fixed percolation configuration, the prescribed source constraints implies that the sites x_ℓ for which $x_1 \longleftrightarrow x_\ell$ have **labels of alternating parity** due to the planarity of the graph. Thus

$$\sum_{\ell=2}^{2n} (-1)^\ell\, \mathbb{I}[x_1 \longleftrightarrow x_\ell] = 1\,.$$

Part II. Finite-range interactions

Heuristic in the case of finite-range interactions

Let us focus on the four-point function. The representation in random-current still works, so that it would be sufficient to study intersections.



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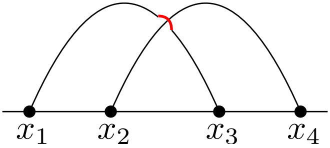

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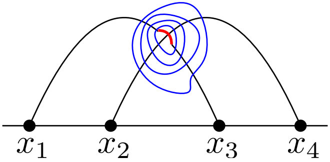

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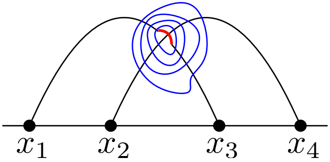

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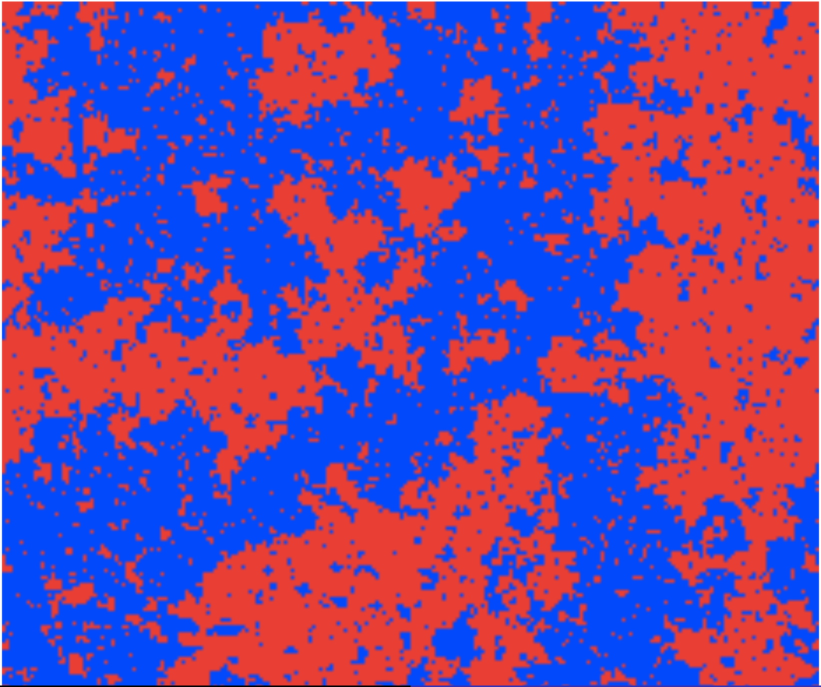
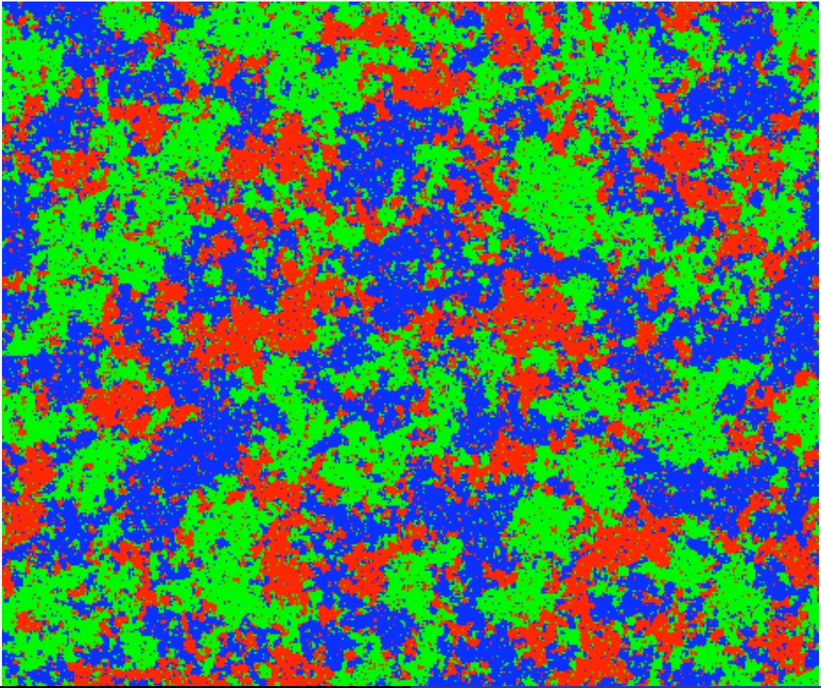
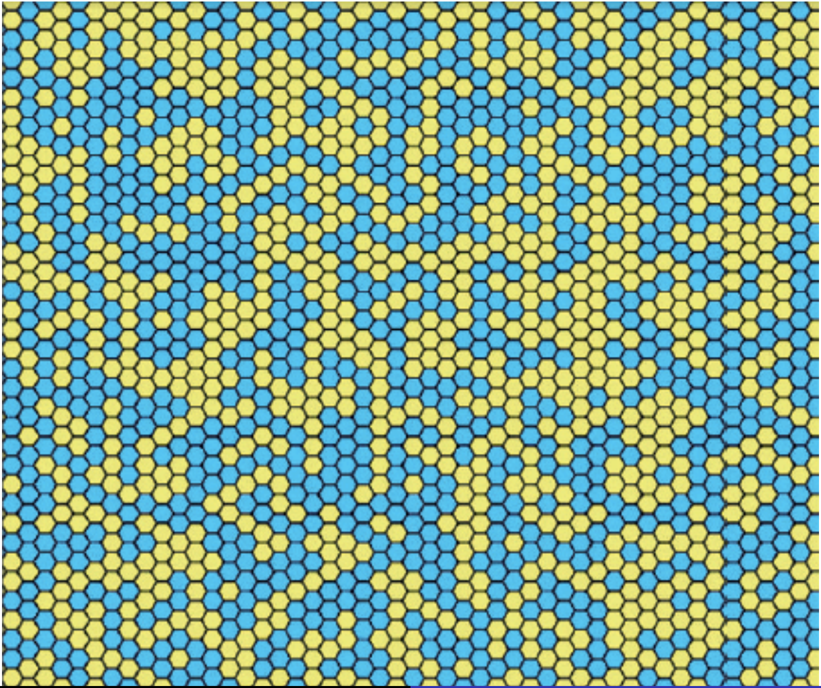
Theorem (Aizenman, Duminil-Copin, Tassion, Warzel (2016))

At β_c, the (infinite-volume sourceless) random current contains infinitely many circuits surrounding the origin almost surely.







Three models related to Ising on a finite graph G

In this part, let us assume (to simplify), that $J_{xy} = 1$ if $\{x, y\}$ is an edge of G, and 0 otherwise.

RC percolation. Model of random subgraph of G obtained by taking the trace $\widehat{\mathbf{m}}$ of a sourceless current $\mathbf{m}$ sampled with probability proportional to $w(\mathbf{m})$.

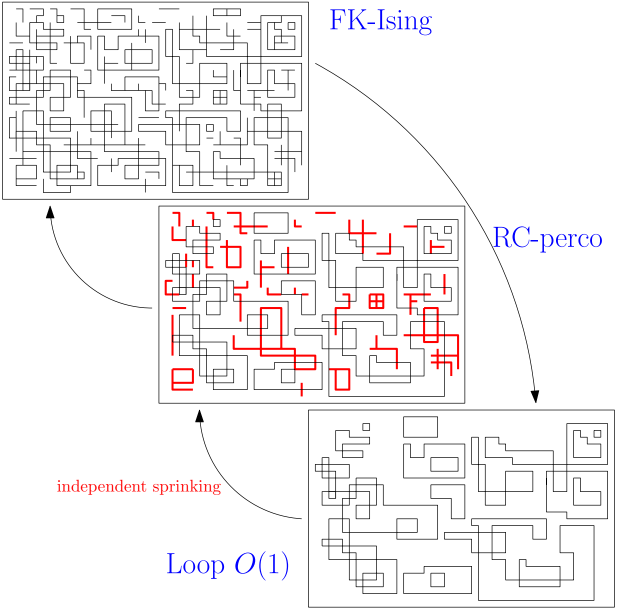
Loop $O(1)$ model. Model of random even subgraph η of G obtained from the **high-temperature expansion** of the model (the probability is proportional to $\tanh(\beta)^{|\eta|}$. For G planar, the loops correspond to **interfaces of the Ising model on** G^* by Kramers-Wannier duality.

FK-Ising percolation. Model of random subgraph ω of G, where

$$\phi(\omega) \ := \ \frac{1}{Z}\Big(\frac{p}{1-p}\Big)^{\#\text{edges in }\omega} q^{\#\text{connected components in }\omega}$$

with $p = 1 - e^{-2\beta}$ and $q = 2$.

Coupling between these models



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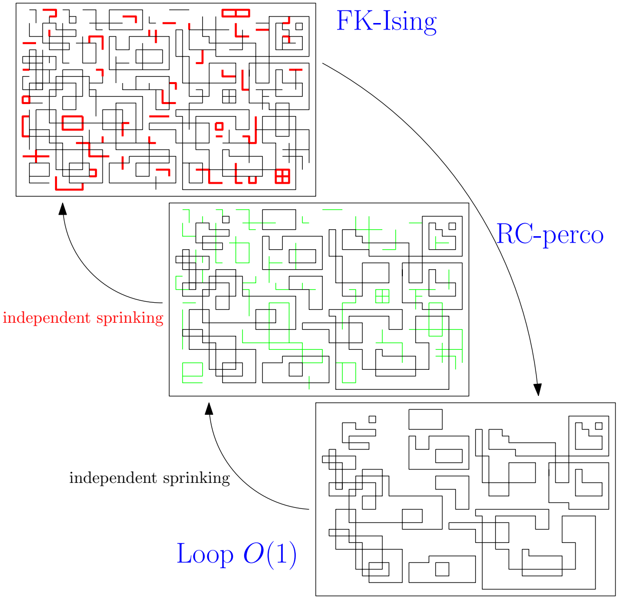

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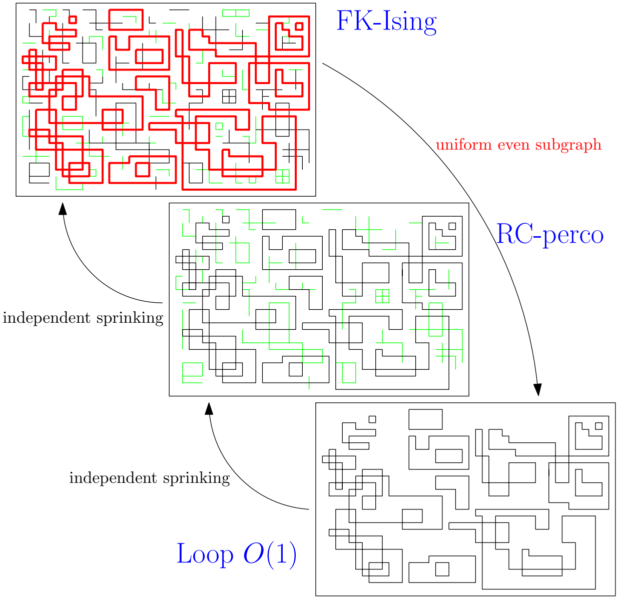

Synergy between models

1. RC percolation. Particularly useful when working with truncated correlation functions, especially because of the **switching lemma**.

2. Loop $O(1)$-model. Rich combinatorial structure due to the constraints on configurations being even subgraphs. Additional switching principles. Also, in the planar case interpretation in terms of interfaces.

3. FK-Ising. FKG measure: the model is positively associated. In particular, one can prove a bunch of general theorems on the critical phase.

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There are infinitely many circuits around the origin in ω almost surely.

Heuristic proof for finite-range interactions

The two previous theorems on FK-Ising show that there are infinitely many distinct connected components surrounding the origin almost surely.

We now wish to prove that in a uniformly chosen even subgraph, there is infinitely many circuits surrounding the origin.

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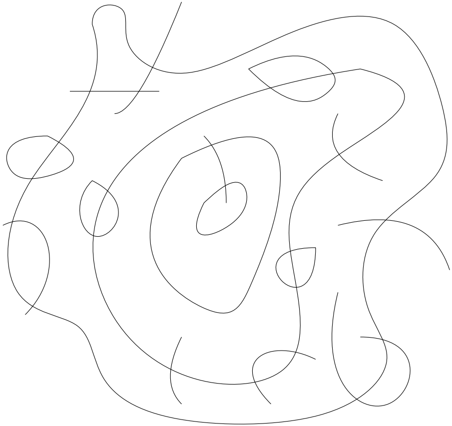

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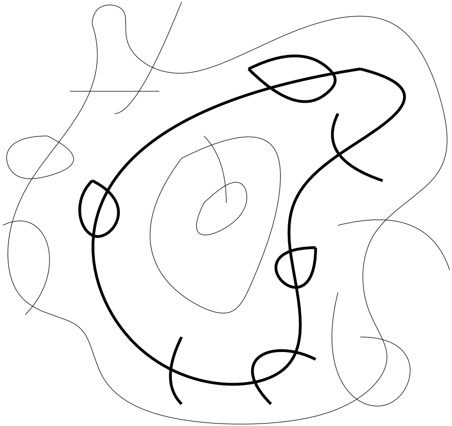

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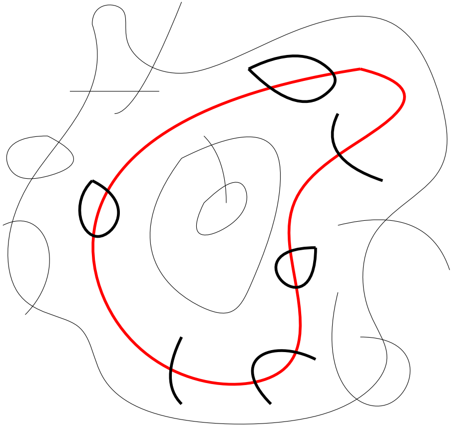

Conclusion

One step towards universality: the behavior at criticality is expected to be independent of the local definition. It should depend on

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- The coupling generalizes to Ashkin-Teller models and has new applications there.