Introduction and the proof

The Atiyah-Patodi-Singer index theorem (APS theorem) is used in this text as a pivot (or maybe an excuse) to discuss some aspects of geometry and analysis on manifolds with boundary. This volume does *not* contain a general treatment of index theorems even though they are amongst the most basic analytic-geometric results one can find. The power of such theorems in applications largely lies in their simplicity and generality. In particular the statement of the APS theorem is quite simple. In practice a great deal of effort, by many people, has gone into simplifying the proofs. This has lead to, and been accompanied by, a much wider understanding of the analytic framework in which they are centred. In fact, from an analytic perspective, index theorems can be thought of as much as testing grounds, for methods and concepts, as ends in themselves. The Atiyah-Singer theorem, which is the boundaryless precursor to the APS theorem, is intimately connected to the theory of pseudodifferential operators. This volume *is* intended to place the APS theorem in a similar context, the 'b' category and related calculus of b-pseudodifferential operators on a compact manifold with boundary.

The basic approach adopted here is to 'state' and 'prove' the APS theorem immediately, being necessarily superficial on a variety of points. The subsequent nine chapters consist largely in the fleshing out of this proof. Just as the initial discussion is brief, the later treatment is discursive and aims at considerably more than the proof of the index theorem alone. The proof given here is *direct* in two senses. The written proof itself is quite straightforward, given some conceptual background, and in particular the terms in the final formula come out directly in the course of the proof. The model here is Getzler's proof ([35]) of the Atiyah-Singer theorem for Dirac operators on a compact manifold without boundary.

The second sense in which the proof is direct is closely connected to the main thesis of this text. Namely that the APS theorem *is* the Atiyah-Singer theorem in the b-category, which is to say the category of compact manifolds with boundary with metrics having complete cylindrical ends. These metrics are called here (exact) b-metrics. This is by no means a radical position (since it is at least implicit in the original papers) but it is a position taken with some fervour. One consequence of this approach is the suggestion that there are other such theorems, especially on manifolds with corners. It is hoped that the context into which the APS theorem is placed will allow it to be readily understood and, perhaps more importantly, generalized. Of course extensions and generalizations already have been made, see in particular the work of Bismut and Cheeger [16], Cheeger [26], Moscovici and Stanton [69], Müller [70] and Stern [85]; see also Wu [91] and Getzler [36].

A review of the proof below, annotated with references to the intervening chapters to make it complete, can be found in §9.1. Towards the end of this Introduction there is an outline of the content of the later chapters.

1. The Atiyah-Singer index theorem.

Consider the Atiyah-Singer index theorem on a compact manifold without boundary. The version for Dirac operators is necessarily proved along the way to the APS theorem. It can be written in brief ([11])

$$\operatorname{ind}(\eth_E^+) = \int_X \mathrm{AS}\,. \tag{In.1}$$

Here $\eth_E^+$ is a twisted Dirac operator, with coefficient bundle E, on the compact even-dimensional spin manifold X and AS is the Atiyah-Singer integrand. This is the volume part (form component of maximal degree) of the product of a characteristic class on X, the $\widehat{A}$ genus, and the Chern character of E :

$$\mathrm{AS} = \mathrm{Ev}_{\dim X}\left(\widehat{A}(X)\cdot \mathrm{Ch}(E)\right). \tag{In.2}$$

Here $\mathrm{Ev}_{\dim X}$ evaluates a form to the coefficient of the volume form of the manifold which it contains. A fundamental feature of (In.1) is that the left side is analytic in nature and the right side is topological, or geometric. One point in favour of Getzler's proof of the index formula is that it is *not* necessary to understand the properties of (In.2) independently, i.e. the theory of characteristic classes is not needed to derive the formula (although it certainly helps to understand it). The left side of (In.1) is, by definition,

$$\operatorname{ind}(\eth_E^+) = \dim \operatorname{null}(\eth_E^+) - \dim \operatorname{null}(\eth_E^-)$$

where $\eth_E^+$ and $\eth_E^-$ act on $\mathcal{C}^\infty$ sections of the appropriate bundles, $\eth_E^-$ being the adjoint of $\eth_E^+$, and the finite dimensionality of the null spaces follows by ellipticity. The direct proof of (In.1) simplifies the original proof of Atiyah and Singer ([11], [12], [72]) and the modifications by Patodi ([73]), Gilkey ([37]) and Atiyah, Bott and Patodi ([5]). Full treatments of proofs along these lines can be found in Berline, Getzler and Vergne [20], Freed [33], Hörmander [48], Roe [77] and Taylor [88].

2. The Atiyah-Patodi-Singer index theorem.

The APS theorem ([8]-[10]) is a generalization of (In.1) to manifolds with boundary. There are two, complementary, ways of thinking about a compact manifold with boundary. The most familiar way is to think of

X as half of a compact manifold *without* boundary by doubling across the boundary. The other approach is to think of the boundary as *at infinity* or at least impenetrable. The latter approach is the one adopted here, whereas in [8]-[10] both approaches are used and the celebrated Atiyah-Patodi-Singer boundary condition (see the discussion in §3 below) comes from the interplay between them. More precisely this means that

$$\text{(In.3)} \qquad \begin{gathered} \eth_E^+ \text{ is a twisted Dirac operator with} \\ \text{respect to an exact } b\text{-spin structure on } X^{2n}. \end{gathered}$$

The reason that X is assumed to be even-dimensional is that in case $\dim X$ is odd and $\partial X = \emptyset$ the index vanishes. There *is* an index theorem in the odd dimensional case but, for Dirac operators, it is relatively simple. For the moment, the notion of a spin structure is left undefined, as is the Dirac operator associated to it.

The exact b-metrics on a compact manifold with boundary are complete Riemann metrics on the interior which make the neighbourhood of the boundary into an asymptotically cylindrical end. More precisely, an exact b-metric is a Riemann metric which takes the form

$$\text{(In.4)} \qquad g = \left(\frac{dx}{x}\right)^2 + h$$

near ∂X, with h a smooth 2-cotensor which induces a Riemann metric on the boundary and $x \in \mathcal{C}^\infty(X)$ a defining function for the boundary. It is important to emphasize that this notion of a b-metric is taken seriously below. For example, the frame bundle of a metric of this type is a smooth principal bundle up to the boundary. An exact b-spin structure is simply a spin structure for an exact b-metric, i.e. a refinement of the frame bundle to a principal Spin bundle, where $\mathrm{Spin}(2n)$ is the non-trivial double cover of $\mathrm{SO}(2n)$.

One important property of an exact b-spin structure (which exists precisely when a spin structure exists) is that it induces a spin structure on the boundary. The corresponding Dirac operator on the boundary will be denoted $\eth_{0,E}$. A useful assumption, which will be removed later, is

$$\text{(In.5)} \qquad \eth_{0,E} \text{ is invertible.}$$

In fact $\eth_{0,E}$ is elliptic and self-adjoint so (In.5) just means that its null space reduces to $\{0\}$. As a consequence of (In.5), $\eth_E^+$ is Fredholm on its natural domain (the Sobolev space defined by the metric) and the APS theorem states that

$$\text{(In.6)} \qquad \operatorname{ind}(\eth_E^+) = \int_X \mathrm{AS} - \frac{1}{2}\eta(\eth_{0,E}).$$

Here the Atiyah-Singer integrand, AS, is the same as before, manufactured from differential-geometric information in the spin structure and auxiliary bundle by local operations. On the other hand the η-invariant is a global object constructed from $\eth_{0,E}$, so fixed purely in terms of boundary data. In fact it is a spectral invariant of $\eth_{0,E}$. This decomposition into a 'local interior' and a 'global boundary' term is fundamental to the utility of the result.

3. Boundary conditions versus b-geometry.

The boundary of a compact manifold with boundary always has a collar neighbourhood, i.e. a neighbourhood of the form $[0,r)_x \times \partial X$, say for $r > 1$. An exact b-metric (In.4) is (or gives the manifold) a cylindrical end if, on the collar, h is simply the pull-back of a metric on ∂X. Often (although not so often here) the end is considered as unbounded in that $t = \log x$ is introduced as a variable, putting the boundary at $t = -\infty$. The manifold $X_1 = X \setminus ([0,1] \times \partial X)$ is diffeomorphic to X.

The spin bundles ${}^{\pm}S$ on X can be identified on the collar neighbourhood of the boundary, since they can be identified over the boundary, by an isomorphism with the spinor bundle of the boundary. Then (see §3.11) the Dirac operator becomes

$$\eth^+ = M_-^{-1} \cdot \left(x\frac{\partial}{\partial x} + \eth_0 \right) \cdot M_+, \tag{In.7}$$

where $M_\pm$ are the isomorphisms between the spinor bundles ${}^{\pm}S$ on the collar and S_0, the spinor bundle of ∂M and and $\eth_0$ is the Dirac operator on the boundary. The null space of $\eth^+$ acting on distributions on the collar can then be examined in terms of the eigen-decomposition for $\eth_0$, which is self-adjoint. Thus the solutions of $\eth^+ u_+ = 0$ are superpositions of the special solutions

$$u = x^{-z} M_+^{-1} v, \ \eth_0 v = zv.$$

This solution is square-integrable with respect to the metric if and only if $z < 0$.

The APS boundary condition for the Dirac operator restricted to the region $x \geq 1$, i.e. $t \geq 0$ is

$$Q_+ \left(M_+ u_{\restriction x=1} \right) = 0, \tag{In.8}$$

where Q_+ is the orthogonal projection onto the non-negative eigenspace of $\eth_0$. This projector is a pseudodifferential operator and $\eth^+$ with the boundary condition (In.8) can be considered as an elliptic boundary problem. In particular there is an associated Fredholm operator and the analysis can be carried out using the theory of elliptic boundary problems introduced by Calderón ([25]) and developed further by Seeley [81] (see also Boutet de Monvel [21] and Grubb [41]).

With the APS boundary condition, the appearance in (In.6) of the eta invariant is a little less striking, since it represents just a part of the information in the projection Q_+. The condition (In.8) reflects the square-integrability of an extension of the solution into $t < 0$, i.e. to the whole of the original manifold X. It will not be encountered below[1]. As already noted, the invertibility of the operator $\eth^+$, which in general for an exact b-metric is not quite as simple as (In.7), is attacked directly and its generalized inverse and the associated heat kernels are shown to be elements of the appropriate space of b-pseudodifferential operators.

4. Preliminaries to the proof.

Let ${}^{\pm}S$ be the two spinor bundles over the compact, even dimensional, exact b-spin manifold, X. The idea, used already in [5] and in a related manner by McKean and Singer [58] and dating back, in other contexts, at least to Minakshisundarum and Pleijel [68] is to consider the heat kernels

$$\exp(-t\eth_E^-\eth_E^+),\ \exp(-t\eth_E^+\eth_E^-), \tag{In.9}$$

where the Dirac operator is

$$\eth_E^+ : \mathcal{C}^\infty(X; {}^+S\otimes E) \longrightarrow \mathcal{C}^\infty(X; {}^-S\otimes E)$$

and $\eth_E^-$ is its adjoint. Suppose for the moment that $\partial X = \emptyset$ and consider the Atiyah-Singer theorem. Both $\eth_E^-\eth_E^+$ and $\eth_E^+\eth_E^-$ are elliptic, self-adjoint and non-negative so the heat kernels (In.9) are, for $t > 0$, smoothing operators. The fact that 0 is an isolated spectral point of both means that

$$\begin{aligned} &\lim_{t\to\infty} \exp(-t\eth_E^-\eth_E^+) = \pi_{\mathrm{null}(\eth_E^+)} \\ &\lim_{t\to\infty} \exp(-t\eth_E^+\eth_E^-) = \pi_{\mathrm{null}(\eth_E^-)} \end{aligned} \tag{In.10}$$

where π_N is orthogonal projection onto the finite dimensional subspace $N \subset L^2(X; L)$, for the appropriate bundle L. The convergence in (In.10) is exponential, within smoothing operators. The trace functional, just the sum of the eigenvalues of a finite rank operator, extends continuously to smoothing operators, so from (In.10) it follows that

$$\lim_{t\to\infty} \mathrm{Tr}[\exp(-t\eth_E^-\eth_E^+) - \exp(-t\eth_E^+\eth_E^-)] = \mathrm{ind}(\eth_E^+). \tag{In.11}$$

[1] The expunging of the APS boundary condition, in explicit form, is a 'feature' of this proof which is fundamental, although not universally welcomed.

The extension of the trace to smoothing operators is described by Lidskii's theorem. Namely smoothing operators are those with $\mathcal{C}^\infty$ Schwartz kernels ([79]), i.e. they can be written as integral operators

$$Ku(x) = \int_X K(x, x')u(x') \text{ with } K \in \mathcal{C}^\infty.$$

Then

$$\operatorname{Tr} K = \int_X K(x, x). \tag{In.12}$$

The single most important property of the trace functional is that it vanishes on commutators:

$$\operatorname{Tr}[K_1, K_2] = 0, \tag{In.13}$$

as follows readily from (In.12). This remains true if K_1 is a differential operator, provided K_2 is smoothing.

At least formally consider (since the operators act on different bundles)

$$\begin{aligned}
&\frac{d}{dt}\big(\exp(-t\eth_E^-\eth_E^+) - \exp(-t\eth_E^+\eth_E^-)\big) \\
&= -\big(\eth_E^-\eth_E^+\exp(-t\eth_E^-\eth_E^+) - \exp(-t\eth_E^+\eth_E^-)\eth_E^+\eth_E^-\big) \\
&= -[\eth_E^-, \eth_E^+\exp(-t\eth_E^-\eth_E^+)].
\end{aligned} \tag{In.14}$$

Here the identity

$$\exp(-t\eth_E^+\eth_E^-)\eth_E^+ = \eth_E^+\exp(-t\eth_E^-\eth_E^+), \tag{In.15}$$

which follows from the uniqueness of solutions to the heat equation, has been used. With the trace taken in (In.14), (In.13) and (In.11) together give the remarkable identity of McKean and Singer

$$\operatorname{ind}(\eth_E^+) = \operatorname{Tr}\left[\exp(-t\eth_E^-\eth_E^+) - \exp(-t\eth_E^+\eth_E^-)\right] \ \forall\ t > 0. \tag{In.16}$$

The formulæ (In.1) and (In.2) arise from a clear understanding of the behaviour of the heat kernels as $t \downarrow 0$, i.e. from the *local index theorem* (proved by Gilkey [37] and Patodi [72]):

$$\operatorname{AS}(x) = \lim_{t\downarrow 0}\operatorname{tr}\big(\exp(-t\eth_E^-\eth_E^+) - \exp(-t\eth_E^+\eth_E^-)\big)(x, x), \tag{In.17}$$

where the 'little' trace, tr, is just the trace functional on the bundles ${}^{\pm}S \otimes E$. The direct proof of (In.17) is Getzler's rescaling argument. The final formula (In.1) arises by applying (In.12) to (In.17).

So, to prove (In.6), in which the η-term should appear as a *defect*, it is natural to look at the heat kernels (In.9) when $\partial X \neq \emptyset$. The *fundamental problem* with the generalization of the proof outlined above is that, when $\partial X \neq \emptyset$ in the 'b-' setting,

$$\exp(-t\eth_E^-\eth_E^+),\ \exp(-t\eth_E^+\eth_E^-) \text{ are not trace class.}$$

Indeed the Atiyah-Patodi-Singer boundary condition was introduced to replace these by trace class operators. There is however a direct generalization of the statement that these exponentials are smoothing operators, which they still are in the interior. Namely there is a calculus of pseudodifferential operators ([61], [64], [47, §18.3], [66]), denoted here $\Psi_b^m(X; L_1, L_2)$ for any bundles L_1, L_2, which captures appropriate uniformity of the kernels up to the boundary. The assumption (In.3) implies that the Dirac operator is in the corresponding space of differential operators

$$\eth_E^+ \in \operatorname{Diff}_b^1(X; {}^+S \otimes E, {}^-S \otimes E) \subset \Psi_b^1(X; {}^+S \otimes E, {}^-S \otimes E);$$

it is elliptic. It follows from constructions essentially the same as in the standard case that

$$\text{(In.18)} \qquad \begin{matrix} \exp(-t\eth_E^-\eth_E^+) \in \Psi_b^{-\infty}(X; {}^+S \otimes E) \\ \exp(-t\eth_E^+\eth_E^-) \in \Psi_b^{-\infty}(X; {}^-S \otimes E) \end{matrix} \quad \text{in } t > 0.$$

As already noted, these conditions do *not* mean that the operators are trace class. Despite this there is an *extension* of the trace functional to a linear functional

$$\text{b-Tr}_\nu : \Psi_b^{-\infty}(X; L) \longrightarrow \mathbb{C}.$$

This extension depends on ν, a trivialization of the normal bundle to ∂X, and is defined simply by regularization of (In.12), as in the work of Hadamard [43]. If $x \in \mathcal{C}^\infty(X)$ is a defining function with $dx \cdot \nu = 1$, the b-trace is

$$\text{(In.19)} \qquad \text{b-Tr}_\nu(K) = \lim_{\epsilon \downarrow 0} \left[\int_{x > \epsilon} \operatorname{tr} K(x,x) + \log \epsilon \cdot \widetilde{\operatorname{Tr}}(K) \right].$$

The logarithmic term removed in (In.19) is precisely what is needed to regularize the integral and this fixes the coefficient $\widetilde{\operatorname{Tr}}(K)$.

The appearance of the defect in the index formula is directly related to the failure of (In.13) for the b-trace. There is an algebra homomorphism in the calculus

$$\Psi_b^m(X; L_1, L_2) \xrightarrow{(\ )_\partial} \Psi^m(\partial X; L_1, L_2), \tag{In.20}$$

where the image space consists of the pseudodifferential operators on the boundary acting on the restrictions of the bundles to the boundary. This map is defined by restriction:

$$A_\partial u = A\tilde{u}_{\restriction \partial X} \text{ if } \tilde{u} \in \mathcal{C}^\infty(X; L_1),\ \ \tilde{u}_{\restriction \partial X} = u.$$

The homomorphism (In.20) can be extended by noting that the calculus is invariant under conjugation by complex powers of x (a boundary defining function)

$$\Psi_b^m(X; L) \ni A \longleftrightarrow x^{-i\lambda} A x^{i\lambda} \in \Psi_b^m(X; L).$$

Then

$$I(A, \lambda) = \left(x^{-i\lambda} A x^{i\lambda}\right)_\partial \in \Psi^m(\partial X; L) \tag{In.21}$$

is an entire analytic family of pseudodifferential operators, the *indicial family* of A. Moreover

$$K \in \Psi_b^{-\infty}(X; L) \text{ is trace class } \iff I(K, \lambda) \equiv 0$$

and

$$I(K, \lambda) \equiv 0 \Longrightarrow \text{b-Tr}_\nu(K) = \text{Tr}(K).$$

The coefficient of the singular term in (In.19) is actually given by

$$\widetilde{\text{Tr}}(K) = \frac{1}{2\pi} \int_{\mathbb{R}} \text{Tr}\left(I(K, \lambda)\right) d\lambda.$$

The fundamental formula for the b-trace is:

$$\text{b-Tr}_\nu([A, B]) = \frac{i}{2\pi} \int \text{Tr}\big(\partial_\lambda I(A, \lambda) \circ I(B, \lambda)\big)\, d\lambda. \tag{In.22}$$

The integral on the right converges absolutely. This formula follows directly from (In.21), (In.19) and the definition of the b-calculus, i.e. it is elementary.

5. The proof.

First some bald facts, which need to be interpreted slightly but are 'true' enough. In terms of (In.21):

$$\text{(In.23)}\qquad \begin{cases} I(\eth_E^+,\lambda)=M_-^{-1}(i\lambda+\eth_{0,E})M_+\\ I(\eth_E^-,\lambda)=M_+^{-1}(-i\lambda+\eth_{0,E})M_-\\ I(\exp(-t\eth_E^-\eth_E^+),\lambda)=M_+^{-1}\exp(-t(\lambda^2+\eth_{0,E}^2))M_+. \end{cases}$$

Here $M_\pm$ are isomorphisms of ${}^{\pm}S$ restricted to ∂X and S_0, the spinor bundle over ∂X, with $M_+^{-1}M_-$ being Clifford multiplication by idx/x. The identity (In.14) still holds, so now taking b-traces it follows from (In.22) that

$$\text{(In.24)}\qquad \begin{aligned} &\frac{d}{dt}\,\text{b-Tr}_\nu\big(\exp(-t\eth_E^-\eth_E^+)-\exp(-t\eth_E^+\eth_E^-)\big)\\ &=\frac{1}{2\pi i}\int \operatorname{Tr}\left[\partial_\lambda I(\eth_E^-,\lambda)\circ I(\eth_E^+\exp(-t\eth_E^-\eth_E^+),\lambda)\right]d\lambda. \end{aligned}$$

Using the fact that I is a homomorphism and (In.23), the right side of (In.24) can be rewritten

$$\text{(In.25)}\qquad -\frac{1}{2\pi}\int_{-\infty}^{\infty}\operatorname{Tr}\left[(i\lambda+\eth_{0,E})\exp(-t(\lambda^2+\eth_0^2))\right]d\lambda.$$

The λ integral can be carried out, replacing the integrand by its even part and changing variable to $t^{\frac12}\lambda$, to give

$$\text{(In.26)}\qquad -\frac{1}{2\sqrt{\pi}}t^{-\frac12}\operatorname{Tr}\big(\eth_{0,E}\exp(-t\eth_{0,E}^2)\big).$$

As $t\to\infty$ (In.11) still holds in case $\partial X\neq\emptyset$, with Tr replaced by b-Tr$_\nu$ (the limit is independent of ν because the limiting operator is trace class). Similarly Getzler's scaling argument carries over to this setting to give (In.17), uniformly in x. Finally a similar scaling argument applies to (In.26) (as shown by Bismut and Freed [18], [19]). This allows (In.24) to be integrated over $(0,\infty)$ giving the limiting formula:

$$\text{(In.27)}\qquad \begin{aligned} &\lim_{t\to\infty}\text{b-Tr}_\nu\big(\exp(-t\eth_E^-\eth_E^+)-\exp(-t\eth_E^-\eth_E^-)\big)\\ &\qquad-\lim_{t\to0}\text{b-Tr}_\nu\big(\exp(-t\eth_E^-\eth_E^+)-\exp(-t\eth_E^+\eth_E^-)\big)\\ &=-\frac{1}{2\sqrt{\pi}}\int_0^\infty t^{-\frac12}\operatorname{Tr}\left(\eth_{0,E}\exp(-t\eth_{0,E}^2)\right)dt. \end{aligned}$$

This then gives (In.6) provided the eta invariant is *defined* to be

$$\eta(\eth_{0,E}) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-\frac{1}{2}} \operatorname{Tr}\big(\eth_{0,E} \exp(-t\eth_{0,E}^2)\big)\, dt, \tag{In.28}$$

where absolute convergence follows from the scaling argument. This is one of the 'standard formulæ' for the η-invariant, so the 'proof' is complete.

6. Weighting.

In this setting there is no explicit boundary condition on the Dirac operator $\eth_E^+$. Rather it is, precisely when (In.5) holds, a Fredholm operator on the Sobolev space fixed by the metric. In fact all exact b-metrics are quasi-isometric, so these Sobolev spaces are intrinsic to the compact manifold with boundary and are denoted $H_b^m(X; L)$, for sections of a vector bundle L.

To prove the APS theorem it is illuminating to embed the index problem in a one-parameter family. Namely the Sobolev spaces extend to weighted spaces, $x^s H_b^m(X; L)$, for $s \in \mathbb{R}$, where $x \in \mathcal{C}^\infty(X)$ is a defining function for the boundary. Then

$$\begin{gathered} \eth_E^+ \colon x^s H_b^1(X; {}^+S \otimes E) \longrightarrow x^s H_b^0(X; {}^-S \otimes E) \\ \text{is Fredholm } \iff -s \text{ is not an eigenvalue of } \eth_{0,E}. \end{gathered} \tag{In.29}$$

The eigenvalues of $\eth_{0,E}$ form a discrete subset $\operatorname{spec}(\eth_{0,E}) \subset \mathbb{R}$, unbounded above and below, so $\operatorname{ind}_s(\eth_E^+)$ is defined for $-s \in \mathbb{R} \setminus \operatorname{spec}(\eth_{0,E})$. It is convenient to extend the definition of the index, even to the case that the operator is not Fredholm by setting

$$\widetilde{\operatorname{ind}}_s(\eth_E^+) = \lim_{\epsilon \downarrow 0} \frac{1}{2} \left[\operatorname{ind}_{s-\epsilon}(\eth_E^+) + \operatorname{ind}_{s+\epsilon}(\eth_E^+) \right]. \tag{In.30}$$

The parameter, s, can be absorbed into the operator by observing that the weighting factor, x^s, can be treated as 'rescaling' (in the sense of Chapter 8) of the coefficient bundle E to a bundle $E(s)$.

Thus the conjugated operator, $\eth_E^+(s) = x^{-s}\eth_E^+ x^s$, is again the positive part of a (twisted) Dirac operator, however the new total Dirac operator is not self-adjoint; its negative part is $\eth_E^-(s) = x^{-s}\eth_E^- x^s$. All of the discussion above applies, provided $\eth_E^-$ is replaced throughout by $(\eth_E^+)^*$, except for the local index theorem, which no longer holds. The Atiyah-Singer integrand can still be defined as

$$\begin{gathered} \operatorname{AS}(s) = \text{The constant term as } t \downarrow 0 \text{ in} \\ \operatorname{tr}\big(\exp(-t\eth_E^-\eth_E^+) - \exp(-t\eth_E^+\eth_E^-)\big)(x, x) \end{gathered} \tag{In.31}$$

and similarly the modified eta invariant[2] is given by the regularized integral

$$\text{(In.32)}\quad \eta_s(\eth_{0,E}) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-\frac{1}{2}} \operatorname{Tr}\left([\eth_{0,E}+s]\exp(-t[\eth_{0,E}+s]^2)\right) dt,\ \ s\in\mathbb{R}.$$

The meaning here is the same as in (In.31): as $t\downarrow 0$ the integral

$$\frac{1}{\sqrt{\pi}} \int_\epsilon^\infty t^{-\frac{1}{2}} \operatorname{Tr}\left([\eth_{0,E}+s]\exp(-t[\eth_{0,E}+s]^2)\right) dt$$

has an expansion in powers of ϵ and $\eta_s(\eth_{0,E})$ is the coefficient of the constant term. The APS theorem then can be written in weighted form

$$\text{(In.33)}\qquad \widetilde{\operatorname{ind}}_s(\eth_E^+) = \int_X \operatorname{AS}(s) - \frac{1}{2}\eta_s(\eth_{0,E}) \quad \forall\ s\in\mathbb{R}.$$

The proof outlined above only applies directly in the Fredholm case, when $s\notin -\operatorname{spec}(\eth_{0,E})$. However only the discussion as $t\to\infty$ needs to be modified to give (In.33) in general.

In fact $\widetilde{\operatorname{ind}}_s(\eth_E^+)$ is locally constant on the open set of Fredholm values of s. Its jump at a point $s\in-\operatorname{spec}(\eth_{0,E})$ can be computed using the relative index theorem discussed in §6.2 or from the direct analysis of $\eta_s(\eth_{0,E})$ in §8.14:

$$\widetilde{\operatorname{ind}}_s(\eth_{0,E}) = \begin{cases} \lim\limits_{r\downarrow s}\operatorname{ind}_r(\eth_{0,E}) + \frac{1}{2}\dim\operatorname{null}(\eth_{0,E}+s) \\ \lim\limits_{r\uparrow s}\operatorname{ind}_r(\eth_{0,E}) - \frac{1}{2}\dim\operatorname{null}(\eth_{0,E}+s). \end{cases}$$

This allows the value of $\widetilde{\operatorname{ind}}_0(\eth_E^+)$ to be computed explicitly as

$$\widetilde{\operatorname{ind}}_0(\eth_E^+) = \dim\operatorname{null}(\eth_E^+) - \dim\operatorname{null}_-(\eth_E^-) + \frac{1}{2}\dim\operatorname{null}(\eth_{0,E}).$$

Here $\operatorname{null}(\eth_E^+)$ is the null space of $\eth_E^+$ on $L^2(X;{}^+S\otimes E)$ and

$$\operatorname{null}_-(\eth_E^-) = \bigcap_{s<0}\left\{u\in x^sL^2(X;{}^-S\otimes E);\eth_E^-u=0\right\}$$

[2] This is not to be confused with the eta function of $\eth_{0,E}$ which is discussed in Chapter 9.

is the 'extended L^2 null space.' This gives the familiar form of the index theorem in general, from [8], even when (In.5) is not valid:

$$\dim \mathrm{null}(\eth_E^+) - \dim \mathrm{null}_-(\eth_E^-) = \int_X \mathrm{AS} - \frac{1}{2}\left[\eta(\eth_{0,E}) + \dim \mathrm{null}(\eth_{0,E})\right].$$

The integer on the left, which is just the index on $x^s L^2(X; {}^+S \otimes E)$ for small $s > 0$, is sometimes called the extended L^2 index.

The absence of the local index theorem in the weighted case means that the form of the integrand, $\mathrm{AS}(s)$, in (In.33) cannot be so easily computed. However it is a polynomial in s. The removal of the non-constant terms allows the general formula to be recast as

$$\text{(In.34).} \qquad \widetilde{\mathrm{ind}}_s(\eth_E^+) = \int_X \mathrm{AS} - \frac{1}{2}\eta(\eth_{0,E}) - \widetilde{N}(\eth_{0,E}, s).$$

Here $\widetilde{N}(\eth_{0,E}, 0) = 0$ and for $s \neq 0$

$$\text{(In.35)} \quad \mathrm{sgn}(s) \times \widetilde{N}(\eth_{0,E}, s) = \text{Number of eigenvalues of } -\eth_{0,E} \text{ in } [0, s],$$

where eigenvalues are counted with their multiplicity and an eigenvalue at an endpoint of $[0, s]$ is counted with half its multiplicity.

7. Outline.

As already noted the remaining chapters are intended to place the proof outlined above on a firm basis and in context. First, in Chapter 1, the one-dimensional case, or rather analogue, of the theorem is discussed, although it is not proved. This discussion is not used later but serves to indicate the different ways of viewing a cylindrical end and introduces the power law behaviour of solutions, and fundamental solutions, which underlies the later analysis. Chapter 2 consists of a brief introduction to Riemannian geometry, the Levi-Civita connection and Riemann curvature tensor, presented in order that the extension to b-metrics should be straightforward. Again the point of view taken is that these metrics correspond in the category of compact manifolds with boundary to Riemann metrics in the boundaryless case. For example, they are fibre metrics on a vector bundle, the b-tangent bundle, which is *not* quite the ordinary tangent bundle but is a perfectly satisfactory replacement for it. The notion of a b-differential operator is introduced, as is the notion of ellipticity in this setting. In Chapter 3 the discussion is extended to examine the Clifford algebra and spin structures. It is shown that the Dirac operator associated to an exact b-spin structure is an elliptic b-differential operator.

The analytic part of the investigation begins, very geometrically, in Chapter 4, with a discussion of the b-stretched product of manifolds with boundary. This is the replacement for the ordinary product that it is convenient to use in the inversion of elliptic b-differential operators and it leads directly to the definition and basic properties of the small b-calculus. In particular the normal homomorphism underlying (In.23) and the product formula are then derived. As noted in the discussion of the composition formula, there is a more elegant, and general, approach using somewhat more differential-geometric machinery (see [63]). Some parts of this approach are introduced in the exercises but it is eschewed here in favour of a more elementary treatment. An effort is made to emphasize the structural properties of the b-calculus. One important feature is the b-trace functional. The commutator identity for this functional, (In.22), plays an important rôle in the proof in that it replaces otherwise cumbersome manipulations of the heat kernel on the cylindrical end, as carried out in [8] and in other versions of the theorem such as [85].

The small calculus of b-pseudodifferential operators reduces to the ordinary calculus of pseudodifferential operators when the compact manifold has no boundary. Philosophically there are two main uses for the ordinary calculus. It is used as an investigative tool (in microlocal analysis) and also to invert elliptic operators. The fact that the same space of operators serves both purposes, when $\partial X = \emptyset$, is somewhat fortuitous. For a manifold with boundary this is no longer the case and to invert elliptic b-differential operators it is necessary to enlarge the calculus. For this purpose, both the 'calculus with bounds' and the 'full calculus' are introduced in Chapter 5. Here the additional boundary terms which appear in the (generalized) inverse are described. The full calculus is applied to the examination of the mapping, and especially Fredholm, properties of elliptic b-differential operators. In Chapter 6 the calculus is further used to establish the relative index theorem and to describe the holomorphy properties of the resolvent family of a self-adjoint operator of second order. The boundary behaviour of the resolvent is also related to scattering theory. As an application of the relative index theorem, using an idea of Gromov and Shubin (see [40]), the Riemann-Roch theorem for surfaces is deduced. This chapter also contains a Hodge theoretic discussion (of course from the point of view of b-metrics) of the cohomology of a compact manifold with boundary.

In Chapter 7 the heat kernel of a second order operator is described. This is done in a manner consistent with the treatment of the b-calculus, i.e. using a blown-up 'heat space' to define the class of admissible kernels. This approach is very similar to the calculi described by Beals and Greiner in [14], by Taylor in [89] and more recently in [31]. Here the geometric structure is made explicit and this has the important consequence that the melding of the heat and b-calculi, to yield a detailed description of the heat kernel of a b-differential operator (such as $\eth_E^- \eth_E^+$) is straightforward. The discussion of the resolvent in Chapter 6 is used to analyze the long-time behaviour of the heat kernel.

Getzler's rescaling argument is formalized in Chapter 8 in the notion of the rescaling, at a boundary hypersurface, of a vector bundle. The weighting of the Sobolev spaces in (In.29) and the b-tangent bundle are both examples of this general procedure. The local index theorem then follows directly from this rescaling, the fundamental observation of Berezin and Patodi on the structure of the supertrace functional on the spin bundle, Lichnerowicz' formula for the difference between the Dirac and connection Laplacians and a generalization of Mehler's formula for the heat kernel of the harmonic oscillator, found by Getzler.

Finally in Chapter 9 the proof of the APS theorem outlined above is reviewed and completed by annotation with references to the intervening material. In fact the theorem is actually proved in the wider context of the Dirac operators on Hermitian Clifford modules (with graded unitary Clifford b-connections) on manifolds with exact b-metrics. The application to the signature formula given in [8] is then explained. It is also shown how the application of the b-calculus allows many of the standard analytic-geometric objects, such as the zeta function, the eta invariant and the Ray-Singer analytic torsion to be transferred to the b-category.

Chapter 1. Ordinary differential operators

The basic analytic tool developed below to carry out the proof of the APS theorem is the calculus of b-pseudodifferential operators. This allows the mapping, especially Fredholm, and spectral properties of $\eth_E^+$ to be readily understood. As motivation for the analytic part of the discussion the one-dimensional case will first be considered, although the result is not proved in detail. This case is 'easy' for many reasons, not least because the dimension is odd, which means there is no interior contribution to the index, and the boundary dimension is zero, so the boundary operator is a matrix, i.e. has finite rank. However, from an analytic point of view the one-dimensional cases serves as quite a good guide to the general case.

1.1. Operators and coordinates.

In one dimension there is *no* spin structure to be concerned about. The only connected one-dimensional compact manifold with non-trivial boundary is the interval $X = [0,1]$. In particular all bundles are trivial. Of course one should bear in mind that the boundary has two components. So consider a first-order linear differential operator acting on k functions

$$P = A(x)\frac{d}{dx} + B(x), \quad A, B \in \mathcal{C}^\infty([0,1]; M_{\mathbb{C}}(k)), \tag{1.1}$$

where $M_{\mathbb{C}}(k) \simeq \mathbb{C}^{k^2}$ is the algebra of $k \times k$ complex-valued matrices.

The operator should be *elliptic* in the interior, so $\det(A(x)) \neq 0$ for $x \in (0,1)$. It should also be of 'b' type at the end-points. This means that P should have regular-singular points at 0 and 1 :

$$A(x) = x(1-x)E(x), \quad E \in \mathcal{C}^\infty([0,1]; M_{\mathbb{C}}(k)),\ \det E \neq 0 \text{ on } [0,1]. \tag{1.2}$$

To analyze the *index* of P consider the adjoint, P^*, and set

$$\operatorname{ind}(P) = \dim \operatorname{null}(P) - \dim \operatorname{null}(P^*) \tag{1.3}$$

or find some space on which P is *Fredholm*:

a) $P : H_1 \longrightarrow H_2$ continuous, H_1, H_2 Hilbert spaces
b) $\operatorname{null}(P) \subset H_1$ finite dimensional
c) $\operatorname{range}(P) \subset H_2$ closed
d) $\operatorname{range}(P)^\perp \subset H_2$ finite dimensional

and then set

$$\operatorname{ind}(P) = \dim \operatorname{null}(P) - \dim(\operatorname{range}(P)^\perp). \tag{1.4}$$

If P is Fredholm these two definitions of the index are the same.

What is a reasonable space on which P can be expected to be Fredholm? Consider the simple case

$$P_c = x(1-x)\frac{d}{dx} + c, \quad c \in \mathbb{C}. \tag{1.5}$$

There are two transformations of the independent variable which yield even simpler operators. First

$$t = \frac{x}{1-x} : [0,\infty) \ni t \longmapsto x = \frac{t}{1+t} \in [0,1) \tag{1.6}$$

and then $s = \log t = \log x - \log(1-x)$, $(-\infty,\infty) \ni s \longmapsto t = e^s \in (0,\infty)$. Notice that

$$\frac{dt}{dx} = \frac{1}{(1-x)^2} = (1+t)^2, \ \frac{dt}{ds} = e^s = t,$$

so

$$x(1-x)\frac{d}{dx} = \frac{x}{1-x}\frac{d}{dt} = t\frac{d}{dt}$$
$$x(1-x)\frac{d}{dx} = \frac{d}{ds}.$$

Thus P_c in (1.5) becomes

$$P_c = t\frac{d}{dt} + c, \ \mathbb{R}^+\text{-invariant on } (0,\infty)$$
$$P_c = \frac{d}{ds} + c, \ \text{translation-invariant on } \mathbb{R}.$$

Certainly then P_c is easy to analyze. Acting on any reasonable class of functions, distributions or even hyperfunctions, P_c has at most a one-dimensional null space, given in the three coordinates systems by

$$P_c u = 0 \Longrightarrow u = \begin{cases} ax^{-c}(1-x)^c \\ at^{-c} \\ ae^{-cs} \end{cases} \quad a \in \mathbb{C}. \tag{1.7}$$

Then the only question is whether or not this solution is in the domain of P_c.

Suppose P_c^* is taken to be the adjoint with respect to Lebesgue measure, $|ds|$, on $\mathbb{R}$

$$P_c^* = -\frac{d}{ds} + \bar{c}.$$

Thus (1.7) applies to $P_c^* = -P_{-\bar{c}}$. It is reasonable to take the *domain* of P_c to be

$$H^1(\mathbb{R}) = \{u \in L^2(\mathbb{R}); \frac{d}{ds}u \in L^2(\mathbb{R})\}, \tag{1.8}$$

the standard Sobolev space.

EXERCISE 1.1. Check that if the domain of P_c^* is defined by

$$\operatorname{Dom}(P_c^*) = \{u \in L^2(\mathbb{R}); H^1(\mathbb{R}) \ni v \longmapsto \int u\overline{P_c v}ds$$
$$\text{extends by continuity to } L^2(\mathbb{R})\}$$

then $\operatorname{Dom}(P_c^*) = H^1(\mathbb{R})$.

1.2. Index.

Now notice that

$$\exp(-cs) \notin L^2(\mathbb{R}) \qquad \forall\ c \in \mathbb{C}, \tag{1.9}$$

since the exponential is *always* too large in one direction or the other (or both if $c \in i\mathbb{R}$). Thus, with domain (1.8) and definition (1.3), it is always the case that

$$\operatorname{ind}(P_c) = 0, \tag{1.10}$$

which is not too interesting!

The constant in P_c can be changed by conjugating by an exponential

$$e^{-as}(\frac{d}{ds} + c)e^{as}u = (\frac{d}{ds} + (a + c))u. \tag{1.11}$$

Since the function e^{as} is real when a is, the adjoint changes to

$$e^{as}(-\frac{d}{ds} + \bar{c})e^{-as} = (-\frac{d}{ds} + (\bar{c} + a)).$$

This corresponds to replacing the Sobolev space (1.8) by the exponentially weighted space

$$e^{as}H^1(\mathbb{R}) = \{u \in L^2_{\text{loc}}(\mathbb{R}); e^{-as}u \in H^1(\mathbb{R})\},$$

where $L^2_{\text{loc}}(\mathbb{R})$ is the space of locally square-integrable functions on $\mathbb{R}$. Certainly (1.10) still holds on these spaces, but somehow not quite for the 'same' reason in that the part of infinity which causes (1.9) may have changed. This suggest that a less trivial result may follow by looking at spaces which are *weighted* differently at the *two* infinities, i.e. boundary points.

Recall that $H^1_{\text{loc}}(0, 1)$ is the space of locally square-integrable functions on $(0, 1)$ with first derivative, in the distributional sense, also given by a locally square-integrable function.

DEFINITION 1.2. For $\alpha, \beta \in \mathbb{R}$ set

$$x^\alpha(1-x)^\beta H_b^1([0,1])$$
$$= \Big\{ u \in H^1_{\mathrm{loc}}((0,1)); \int_0^1 \Big| x^{-\alpha}(1-x)^{-\beta} u \Big|^2 \frac{dx}{x(1-x)} < \infty,$$
$$\int_0^1 \Big| x^{-\alpha}(1-x)^{-\beta} \big(x(1-x)\frac{du}{dx}\big) \Big|^2 \frac{dx}{x(1-x)} < \infty \Big\}.$$

On passing to the variable s on $\mathbb{R}$, these conditions can be written in terms of $w \in H^1_{\mathrm{loc}}(\mathbb{R})$ as the requirements

$$\int_{-\infty}^0 \Big| e^{-\alpha s} w \Big|^2 ds, \ \int_0^\infty \Big| e^{\beta s} w \Big|^2 ds < \infty$$
$$\int_{-\infty}^0 \Big| e^{-\alpha s} \frac{dw}{ds} \Big|^2 ds, \ \int_0^\infty \Big| e^{\beta s} \frac{dw}{ds} \Big|^2 ds < \infty. \tag{1.12}$$

Thus in terms of the variable s these spaces are exponentially weighted at infinity, with different weights.

EXERCISE 1.3. Check that in terms of the variable t the spaces in Definition 1.2 become

$$x^\alpha(1-x)^\beta H_b^1([0,1]) \longleftrightarrow$$
$$\Big\{ v \in H^1_{\mathrm{loc}}((0,\infty)); \int_0^\infty \Big| (\frac{t}{1+t})^{-\alpha}(1+t)^\beta v \Big|^2 \frac{dt}{t} < \infty,$$
$$\int_0^\infty \Big| (\frac{t}{1+t})^{-\alpha}(1+t)^\beta (t\frac{d}{dt}v) \Big|^2 \frac{dt}{t} < \infty \Big\}.$$

PROPOSITION 1.4. *The operator P_c in (1.5) is Fredholm as an operator*

$$P_c : \ x^\alpha(1-x)^\beta H_b^1([0,1]) \longrightarrow x^\alpha(1-x)^\beta L_b^2([0,1]) \tag{1.13}$$

if and only if

$$\alpha \neq -\operatorname{Re} c, \ \beta \neq \operatorname{Re} c; \tag{1.14}$$

its index is

$$\operatorname{ind}(P_c) = \begin{cases} 1 & \alpha < -\operatorname{Re} c, \ \beta < \operatorname{Re} c \\ 0 & \alpha > -\operatorname{Re} c, \ \beta < \operatorname{Re} c \ \textit{or} \ \alpha < -\operatorname{Re} c, \beta > \operatorname{Re} c \\ -1 & \alpha > -\operatorname{Re} c, \ \beta > \operatorname{Re} c. \end{cases} \tag{1.15}$$

PROOF: Certainly P_c is always a continuous linear map (1.13) as follows directly from the definition of these spaces. To study the Fredholm properties, it is enough to *invert* P and then see how the inverse is affected by the weighting. Using (1.11) the constant c can be removed since

$$(\frac{x}{1-x})^{-c} : \; x^\alpha(1-x)^\beta H^1_b([0,1]) \longleftrightarrow x^{a-\operatorname{Re} c}(1-x)^{b+\operatorname{Re} c} H^1_b([0,1])$$

for any $c \in \mathbb{C}$ and

$$(\frac{x}{1-x})^c P_c (\frac{x}{1-x})^{-c} = x(1-x)\frac{d}{dx}. \tag{1.16}$$

This reduces the problem to the special case $c = 0$.

Now it is convenient to work in the translation-invariant picture, where $P = d/ds$. An inverse is given by integration, say from 0 :

$$Q'g(s) = \int_0^s g(r)dr \Longrightarrow \frac{d}{ds}Q'g(s) = g. \tag{1.17}$$

Suppose $f \in x^\alpha(1-x)^\beta L^2_b([0,1])$ and $g \in L^2_{\mathrm{loc}}(\mathbb{R})$ is its expression in terms of the coordinate s. From (1.12) it follows that

$$\beta < 0 \Longrightarrow \int_0^\infty \left| e^{s\beta} Q'g(s) \right|^2 ds < \infty$$

$$\alpha < 0 \Longrightarrow \int_{-\infty}^0 \left| e^{-s\alpha} Q'g(s) \right|^2 ds < \infty.$$

Thus if Qf is $Q'g$ expressed in terms of the coordinate x,

$$\alpha, \beta < 0 \Longrightarrow Q : \; x^\alpha(1-x)^\beta L^2_b([0,1]) \longrightarrow x^\alpha(1-x)^\beta H^1_b([0,1]).$$

In this case P_c is surjective, so certainly Fredholm. From the definition, (1.4), of the index and the fact that the null space is spanned by

$$1 \in x^\alpha(1-x)^\beta H^1_b([0,1]) \text{ if } \alpha < 0, \; \beta < 0,$$

the validity of the first case in (1.15) follows.

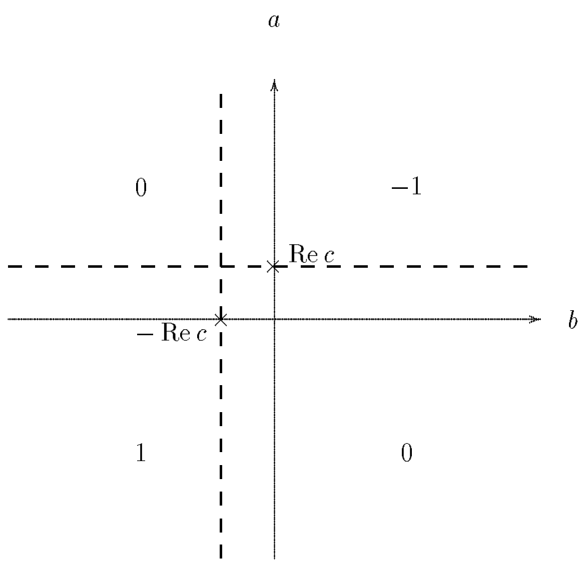

Figure 1. The index of P_c.

Suppose $\beta > 0$, then (1.17) does not give a solution correctly weighted at infinity, unless $\int_0^\infty g(s)ds = 0$. However taking instead

$$Q'g = \int_\infty^s g(r)dr \Longrightarrow \frac{d}{ds}Q'g = g, \tag{1.18}$$

since if $f \in x^\alpha(1-x)^\beta L_b^2$ then g *is* integrable near $s = +\infty$. In this case

$$\alpha < 0,\ \beta > 0 \Longrightarrow P \text{ is surjective, } \operatorname{ind}(P) = 0,$$

since 1 is no longer in $x^\alpha(1-x)^\beta \mathrm{L}_b^2$. The same argument applies if $\alpha > 0$, $\beta < 0$ by replacing x by $1-x$, and hence s by $-s$.

Finally if $\alpha > 0$, $\beta > 0$ then (1.18) is still correctly behaved near infinity but

$$Qf \in x^\alpha(1-x)^\beta L_b^2([0,1]),\ \alpha, \beta > 0 \text{ iff } \int_{-\infty}^{\infty} f(x)\frac{dx}{x(1-x)} = 0.$$

Certainly the constant solution is not in the domain so $\operatorname{ind}(P) = -1$.

This completes the proof of the proposition except for the part of (1.14) which states that P_c is not Fredholm as an operator (1.13) if

$$\alpha = -\operatorname{Re} c \text{ or } \beta = \operatorname{Re} c. \tag{1.19}$$

This is left as an exercise:

Exercise 1.5. Show that if $\alpha = \operatorname{Re} c$ or $\beta = -\operatorname{Re} c$ then P in (1.13) is not Fredholm because the range is *not* closed. [Hint: Find a sequence of functions in $L^2(\mathbb{R})$ of the form du_k/ds, $u_k \in H^1(\mathbb{R})$ such that $du_k/ds \longrightarrow 0$ in L^2 but $\|u_k\|_{L^2} \longrightarrow \infty$. Use this to show that there exists $f \in L^2(\mathbb{R})$ which is in the closure of the range but $f \neq du/ds$, for any $u \in H^1(\mathbb{R})$.]

1.3. General statement.

Proposition 1.4 can be interpreted informally as saying that the operator, P_c, is Fredholm *unless* there is an element of the null space (on distributions) which is *almost* in the domain, but is *not* in the domain. Notice also that in Figure 1 the index increases by 1 (the dimension of this null space) every time one of the lines in (1.19) is crossed downward. These two ideas will reappear in the higher dimensional setting below.

Now consider the extension of this result to the general case, (1.1) subject to (1.2). This is the prototype for the Dirac operator.

Theorem 1.6. *The operator P in (1.1), subject to (1.2), is always a continuous linear operator*

$$P:\ x^{\alpha}(1-x)^{\beta}H^1_b([0,1];\mathbb{C}^k) \longrightarrow x^{\alpha}(1-x)^{\beta}L^2_b([0,1];\mathbb{C}^k),\ \ \alpha,\beta\in\mathbb{R},$$

which is Fredholm if and only if

$$\begin{aligned} &\alpha \neq \operatorname{Re}\lambda \text{ for any eigenvalue } \lambda \text{ of } -E(0)^{-1}B(0)\\ &\beta \neq \operatorname{Re}\lambda \text{ for any eigenvalue } \lambda \text{ of } E(1)^{-1}B(1)\end{aligned} \tag{1.20}$$

and then its index is

$$\operatorname{ind}(P) = -\frac{1}{2}\left(\eta^0_{\alpha} + \eta^1_{\beta}\right), \tag{1.21}$$

where if G^i_{λ} are, for $i = 0, 1$, the eigenspaces with eigenvalue λ of the matrices $(-1)^i(E(i))^{-1}B(i)$,

$$\eta^i_r = \sum_{\operatorname{Re}\lambda > -r} \dim G^i_{\lambda} - \sum_{\operatorname{Re}\lambda < -r} \dim G^i_{\lambda}. \tag{1.22}$$

EXERCISE 1.7. Try to prove this result. It is not too hard, using standard results on solutions of ordinary differential equations, as in [28]. It is also illuminating to follow the lines of the proof outlined in the Introduction in this case.

Consider how the formula (1.21) reduces to (In.6). First note that the Atiyah-Singer integrand, AS, vanishes identically because the dimension is odd. It is worth noting the relationship between (1.22) and (In.28). From (In.23) it is reasonable to expect that

$$\eth_0 \longleftrightarrow E(0)^{-1}B(0).$$

To make this correspondence more exact, suppose that $\eth_0 = E(0)^{-1}B(0)$ is a self-adjoint matrix and $a = 0$. Then

$$\operatorname{tr} \eth_0 \exp(-t\eth_0^2) = \sum_{\text{eigenvalues}} \lambda e^{-t\lambda^2}, \tag{1.23}$$

where the eigenvalues are repeated with their multiplicity. By the assumption (In.5), or equivalently the condition (1.20), 0 should not be an eigenvalue. Then, inserting (1.23) into (In.28) gives

$$\begin{aligned}\eta(\eth_0) &= \frac{1}{\sqrt{\pi}} \sum_{\text{eigenvalues}} \operatorname{sgn}(\lambda) \int_0^\infty e^{-t|\lambda|^2}(t|\lambda|^2)^{-\frac{1}{2}} d(t|\lambda|^2) \\ &= \sum_{\text{eigenvalues}} \operatorname{sgn}(\lambda),\end{aligned}$$

since

$$\int_0^\infty e^{-s} \frac{ds}{s^{\frac{1}{2}}} = \sqrt{\pi}.$$

Thus $\eta(\eth_0) = \eta_0^0$ in terms of (1.22). This shows the relationship between (In.6) and (1.21). It also suggests that the eta invariant measures the *spectral asymmetry* of the operator, i.e. the difference between the number of positive and the number of negative eigenvalues.

EXERCISE 1.8. Check the relationship between (1.21) and (In.33).

1.4. Kernels.

To finish this look at the one-dimensional case, consider again the trivial case (1.5). For $a << 0$, $b >> 0$ the solution operator to P_c is obtained

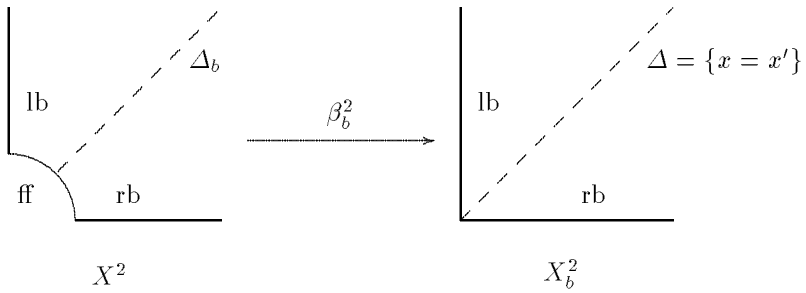

Figure 2. Blow-up of X^2, $X = [0, 1]$.

using the conjugation (1.16), the change of coordinates (1.6) and the integration formula (1.18). Consider this inverse in terms of the compact, x-representation. Then it can be written

$$Qf(x) = \int_1^x f(x') \frac{dx'}{x'(1-x')}.$$

Undoing (1.16) gives the inverse to P_c as

$$Q_c f(x) = \int_1^x \left(\frac{x}{1-x}\right)^c \left(\frac{x'}{1-x'}\right)^{-c} f(x') \frac{dx'}{x'(1-x')}.$$

This can be written as an integral operator

$$Q_c f(x) = \int_0^1 K(x, x') f(x') \frac{dx'}{x'(1-x')},$$

where the Schwartz kernel is

$$K_c(x, x') = -\frac{x^c}{(1-x)^c} \frac{(1-x')^c}{(x')^c} H(x' - x), \tag{1.24}$$

$H(t)$ being the Heaviside function.

Consider the structure of K_c. There are, in principle, five singular terms, with singularities at $x = 0, x = 1, x' = 0, x' = 1$ and $x = x'$ (except that those at $x' = 0$ and $x = 1$ happen to vanish). The singularities are simple power type, except that at the two corners $x = x' = 0$ and $x = x' = 1$

two singularities coincide and things look nasty. It is exactly such kernels that will be analyzed below. To do so, it is convenient to introduce *polar coordinates* around $x = x' = 0$ and $x = x' = 1$.

Actually, to carry out this process of *blowing up*, it is simpler to use the singular coordinates (near $x = x' = 0$)

$$r = x + x',\ \tau = \frac{x - x'}{x + x'}. \tag{1.25}$$

Then $x = \frac{1}{2}r(1 + \tau)$, $x' = \frac{1}{2}r(1 - \tau)$. Inserting this into (1.24) gives

$$K_c(x, x') = \frac{(1+\tau)^c}{(1-\tau)^c} \times H(-\tau) \times \mathcal{C}^\infty \text{ near } x = x' = 0. \tag{1.26}$$

Notice what is accomplished by this maneuver. The kernel now has singularities at two separated surfaces, $\tau = 0$ and $\tau = -1$. This trick is the basis of the b-calculus.

EXERCISE 1.9. Write out the relationship between the coordinates r, τ in (1.25) and polar coordinates ρ, θ where $x = \rho\cos\theta$, $x' = \rho\sin\theta$ and $\theta \in [0, \frac{1}{4}\pi]$. Check that the map $\rho, \theta \longrightarrow (r, \tau)$ is a diffeomorphism from $[0, \infty) \times [0, \frac{1}{4}\pi]$ onto $[0, \infty) \times [-1, 1]$.

EXERCISE 1.10. Find formulæ similar to (1.26) for the Schwartz kernel of a generalized inverse to P_c in the other cases in (1.15).

Chapter 2. Exact b-geometry

Much of this chapter is geometric propaganda. It is intended to convince the reader that there is a 'category' of b-Riemann manifolds in which one can work systematically. This b-geometry can also be thought of as the geometry of manifolds with asymptotically cylindrical ends. There are other geometries which are similar to b-geometry (see [55], [54], [56], [32], [31] and [63] for a general discussion).

Following the definition and discussion of the most basic elements of (exact) b-geometry, the Levi-Civita connection is described *ab initio*. The notion of a b-connection is introduced and its relation to that of an ordinary connection is explained. Finally a brief description of characteristic classes is given. For the reader familiar with differential geometry, the main sections to peruse are §§2.2, 2.3, 2.4, 2.13, 2.16 and 2.17.

2.1. Manifolds.

It is assumed below that the reader is familiar with elementary global differential geometry, i.e. the concept of a manifold. However at various points later, the less familiar notion of a manifold with corners is encountered so, for the sake of clarity, definitions are given here. These have been selected for terseness rather than simplicity or accessibility!

A *topological manifold* of dimension N is a paracompact Hausdorff (connected unless otherwise noted) topological space, X, with the property that each point $p \in X$ is contained in an open set $O \subset X$ which is homeomorphic to $\mathbb{B}^N = \{x \in \mathbb{R}^N; |x| < 1\}$. These open sets, with their maps to $\mathbb{B}^N$, are called coordinate patches.

The algebra of all continuous functions, real-valued unless otherwise stated, is denoted $\mathcal{C}^0(X)$. A subalgebra $\mathcal{F} \subset \mathcal{C}^0(X)$ is said to be a $\mathcal{C}^\infty$ *subalgebra* if for any real-valued $g \in \mathcal{C}^\infty(\mathbb{R}^k)$, for any k, and any elements $f_1, \dots, f_k \in \mathcal{F}$ the continuous function $g(f_1, \dots, f_k) \in \mathcal{F}$. The subalgebra is said to be *local* if it contains each element $g \in \mathcal{C}^0(X)$ which has the property that for every set O_α in some covering of X by open sets there exists $g_\alpha \in \mathcal{F}$ with $g = g_\alpha$ on O_α.

A *manifold* (meaning here always an infinitely differentiable, shortened to $\mathcal{C}^\infty$, manifold) is a topological manifold with a real, local, $\mathcal{C}^\infty$ subalgebra $\mathcal{C}^\infty(X) \subset \mathcal{C}^0(X)$ specified with the following property: X has a covering by open sets O_α, $\alpha \in A$, for each of which there are N elements $f_1^\alpha, \dots, f_N^\alpha \in \mathcal{C}^\infty(X)$ with $F^\alpha = (f_1^\alpha, \dots, f_N^\alpha)$ restricted to O_α making it a coordinate patch and $f \in \mathcal{C}^\infty(X)$ if and only if for each $\alpha \in A$ there exists $g_\alpha \in \mathcal{C}^\infty(\mathbb{B}^N)$ such that $f = g_\alpha \circ F^\alpha$ on O_α.

Exercise 2.1. Show that this definition is equivalent to the standard one involving covering by compatible infinitely differentiable coordinate systems. [Hint: Show that the O_α in the previous paragraph form such a covering. To do so first check that if $g \in \mathcal{C}^\infty_c(\mathbb{B}^N)$ is a $\mathcal{C}^\infty$ function vanishing outside a compact subset of $\mathbb{B}^N$ then for fixed α,

$$f = \begin{cases} g \circ F^\alpha & \text{on } O_\alpha \\ 0 & \text{on } X \setminus O_\alpha \end{cases}$$

is an element of $\mathcal{C}^\infty(X)$ – this argument uses both the $\mathcal{C}^\infty$ algebra and locality properties of $\mathcal{C}^\infty(X)$. Use this in turn to show that the coordinate patches are $\mathcal{C}^\infty$ compatible since, under the transition maps, $\mathcal{C}^\infty$ functions of compact support pull back to be $\mathcal{C}^\infty$.]

Open subsets of manifolds are naturally manifolds. A map $f : X \longrightarrow Y$ between manifolds is smooth, i. e. $\mathcal{C}^\infty$, if $f^*g = g \circ f \in \mathcal{C}^\infty(X)$ whenever $g \in \mathcal{C}^\infty(Y)$. A map is a *diffeomorphism* if it is $\mathcal{C}^\infty$ and has a $\mathcal{C}^\infty$, two-sided, inverse. A finite set of real-valued functions $f_1, \dots, f_k \in \mathcal{C}^\infty(X)$, is *independent* at p if there exist $g_1, \dots, g_{N-k}$, where $N = \dim X$, such that $F = (f_1, \dots, f_k, g_1, \dots, g_{N-k})$ restricts to some neighbourhood of p to a diffeomorphism onto an open subset of $\mathbb{R}^N$.

In the sequel we are most involved with manifolds with boundary. A *topological manifold with boundary* is defined exactly as for a topological manifold, except that each point is only required to have an open neighbourhood homeomorphic to either $\mathbb{B}^N$ or to $\mathbb{B}^N_+ = \{x \in \mathbb{R}^N; |x| < 1, x_1 \geq 0\}$. Since these spaces are not themselves homeomorphic, the two cases are distinct and the subset of X consisting of the points with neighbourhoods homeomorphic to $\mathbb{B}^N_+$ constitutes the boundary, ∂X.

EXERCISE 2.2. Show that the interior, $X \setminus \partial X$, and the boundary of a topological manifold with boundary are both topological manifolds.

Infinitely differentiable manifolds with boundary can be defined quite analogously to the boundaryless case. However we give instead a more extrinsic definition of manifolds with corners and then specialize. A *manifold with corners* (always by implication $\mathcal{C}^\infty$) is a topological manifold with boundary with a local $\mathcal{C}^\infty$ subalgebra $\mathcal{C}^\infty(X) \subset \mathcal{C}^0(X)$ specified with the following property: there is a map into a manifold $\widetilde{X}$, $\iota \colon X \longrightarrow \widetilde{X}$, for which $\mathcal{C}^\infty(X) = \iota^*\mathcal{C}^\infty(\widetilde{X})$ and a finite collection of functions $\rho_i \in \mathcal{C}^\infty(\widetilde{X})$, $i \in I$, for which $\iota(X) = \left\{y \in \widetilde{X}; \rho_i(y) \geq 0 \ \forall\ i \in I\right\}$ and for each $J \subset I$ the ρ_j, for $j \in J$, are independent at each point $p \in \widetilde{X}$ at which they all vanish. The manifold $\widetilde{X}$ is an extension of X.

This definition forces the boundary of a manifold with corners to be a union of embedded hypersurfaces. A direct local coordinate definition (with coordinate patches modeled on $\{x \in \mathbb{R}^N; |x| < 1, x_i \geq 0, i = 1, \dots, k\}$)

does not automatically ensure this. See [63] for a discussion of this point. Smooth maps are defined as before.

EXERCISE 2.3. Show that the product of two manifolds with corners is a manifold with corners.

A *manifold with boundary* is the special case of a manifold with corners when I can be taken to have one element. The boundary of a manifold with boundary is necessarily a manifold without boundary.

A *(real) vector bundle* of rank p over a manifold with corners, Y, consists of a smooth map $f\colon X \longrightarrow Y$ of manifolds with corners where the fibres $f^{-1}(y) \subset X$ have linear structures (of dimension p) which vary smoothly in the sense that the subspace $\mathcal{F} \subset \mathcal{C}^\infty(X)$ of $\mathcal{C}^\infty$ functions which are affine linear on the fibres generates the $\mathcal{C}^\infty$ structure. This last condition means that $\mathcal{C}^\infty(X)$ is the smallest local $\mathcal{C}^\infty$ subalgebra of $\mathcal{C}^0(X)$ containing $\mathcal{F}$ and furthermore $f^*\mathcal{C}^\infty(Y) \subset \mathcal{F}$ is exactly the space of fibre-constant smooth functions.

EXERCISE 2.4. Not only check that this reduces to your favourite definition in case Y (and hence X) has no boundary, but show that such a vector bundle can always be obtained as the restriction of a vector bundle from an extension of Y to a manifold without boundary.

2.2. The b-tangent bundle.

Let X be a compact $\mathcal{C}^\infty$ manifold with boundary. The differential geometry of X will be developed by straightforward extension from the boundaryless case, which is first recalled succinctly. The space X comes equipped with its algebra of $\mathcal{C}^\infty$ functions, $\mathcal{C}^\infty(X)$. Localizing this leads to the cotangent bundle. Thus, if $p \in X$, let

$$\mathcal{I}_p(X) = \{u \in \mathcal{C}^\infty(X); u(p) = 0\}$$

be the ideal of functions vanishing at p and define the cotangent space at p by $T_p^*X = \mathcal{I}_p/\mathcal{I}_p^2$ where $\mathcal{I}_p^2$ is the linear span of products of pairs of elements of $\mathcal{I}_p$. Clearly T_p^*X has a natural linear structure inherited from $\mathcal{I}_p$. In local coordinates $z^1, \dots, z^N$ based at p any element of $\mathcal{I}_p$ is of the form $\sum_{j=1}^N a_j z^j$, modulo $\mathcal{I}_p^2$. The constants a_j therefore give linear coordinates in T_p^*X. The dual space is the tangent space at p, $T_pX = (T_p^*X)^*$. Both the cotangent and the tangent spaces combine to give $\mathcal{C}^\infty$ vector bundles over X, including in the case of a manifold with boundary:

$$TX = \bigsqcup_{p\in X} T_pX, \quad T^*X = \bigsqcup_{p\in X} T_p^*X.$$

Any $\mathcal{C}^\infty$ function on X defines a section, $df : X \longrightarrow T^*X$, of T^*X where $df(p) = [f - f(p)] \in T^*_pX$. The space of *all* $\mathcal{C}^\infty$ sections of T^*X is just the span

$$\mathcal{C}^\infty(X; T^*X) = \operatorname*{span}_{\mathcal{C}^\infty(X)} \{df; f \in \mathcal{C}^\infty(X)\}.$$

Over the boundary of X there is a natural subbundle of TX, namely

$$T\partial X \hookrightarrow T_{\partial X}X = \bigsqcup_{p \in \partial X} T_pX. \tag{2.1}$$

This is defined by noting that the conormal space to the boundary

$$N^*_p\partial X = \{df(p); f \in \mathcal{C}^\infty(X), f_{\restriction \partial X} = 0\} \subset T^*_pX, p \in \partial X$$

is a line in T^*_pX such that $0 \longrightarrow N^*\partial X \longrightarrow T^*_{\partial X}X \longrightarrow T^*\partial X \longrightarrow 0$ is a short exact sequence, where the projection is given by pull-back to the boundary. Then (2.1) follows by duality, with $T_p\partial X$ the annihilator of $N^*_p\partial X$.

The space of all $\mathcal{C}^\infty$ sections of TX is the Lie algebra of vector fields; it acts on $\mathcal{C}^\infty(X)$:

$$\begin{gathered}\mathcal{C}^\infty(X; TX) = \mathcal{V}(X) \ni V,\ V : \mathcal{C}^\infty(X) \longrightarrow \mathcal{C}^\infty(X)\\ Vf(p) = df(p)\big(V(p)\big).\end{gathered}$$

The Lie bracket is fixed by

$$[V, W]f = VWf - WVf \quad \forall\ f \in \mathcal{C}^\infty(X).$$

Now the inclusion (2.1) allows one to define the subspace of vector fields tangent to the boundary

$$\mathcal{V}_b(X) = \big\{V \in \mathcal{V}(X); V_{\restriction \partial X} \in \mathcal{C}^\infty(\partial X; T\partial X)\big\}. \tag{2.2}$$

It is a Lie subalgebra of $\mathcal{V}(X)$. The *fundamental point* leading to *b-geometry* is that $\mathcal{V}_b(X)$ is itself the space of *all* $\mathcal{C}^\infty$ sections of a vector bundle:

$$\mathcal{V}_b(X) = \mathcal{C}^\infty(X; {}^bTX). \tag{2.3}$$

The bundle ${}^bTX$, defined so (2.3) holds, is called the *b-tangent bundle.*

Lemma 2.5. *For each $p \in X$ (including boundary points) define the vector space*

$$ {}^bT_pX = \mathcal{V}_b(X)/\mathcal{I}_p \cdot \mathcal{V}_b(X). \tag{2.4}$$

Then there is a unique vector bundle structure on

$$ {}^bTX = \bigsqcup_{p \in X} {}^bT_pX$$

as a bundle over X such that (2.3) holds in the sense that, under the natural vector bundle map ${}^bTX \longrightarrow TX$, $\mathcal{V}_b(X)$ pulls back to $\mathcal{C}^\infty(X; {}^bTX)$.

PROOF: By definition

$$\mathcal{I}_p \cdot \mathcal{V}_b(X) = \big\{V \in \mathcal{V}_b(X); V = \sum_{\text{finite}} f_j V_j, V_j \in \mathcal{V}_b(X), f_j \in \mathcal{I}_p(X)\big\}.$$

This is certainly a linear space for each $p \in X$. The first thing to check is that the dimension of the quotient, (2.4), is constant. Of course over the interior, essentially by definition,

$${}^bT_pX \simeq T_pX = \mathcal{V}(X)/\mathcal{I}_p \cdot \mathcal{V}(X), \quad p \in X\backslash\partial X.$$

Near any boundary point local coordinates $x, y_1, \dots, y_n$, where $\dim X = n+1$, can always be introduced which are adapted to the boundary, in the sense that

$$x \geq 0, \quad \partial X = \{x = 0\} \text{ near } p$$

and which all vanish at p. Then a $\mathcal{C}^\infty$ vector field can be written in terms of the local derivations

$$\mathcal{V}(X) \ni V \Longrightarrow V_{\restriction O} = a\frac{\partial}{\partial x} + \sum_{j=1}^n b_j \frac{\partial}{\partial y_j},$$

where $O \subset X$ is the coordinate patch and the coefficients are elements of $\mathcal{C}^\infty(O)$. From (2.2) it follows that

$$\begin{aligned} V \in \mathcal{V}_b(X) &\Longrightarrow a_{\restriction \partial X} = 0 \text{ i.e. } a = x\alpha, \text{ with } \alpha \in \mathcal{C}^\infty(X), \\ &\Longrightarrow V = \alpha x \frac{\partial}{\partial x} + \sum_{j=1}^n b_j \frac{\partial}{\partial y_j}. \end{aligned} \tag{2.5}$$

This indeed is the main content of the lemma, that $x\partial/\partial x$ and the $\partial/\partial y_j$, for $j = 1, \dots, n$ form a local basis for ${}^bTX$ in any adapted coordinates (which will just be called coordinates from now on). So ${}^bT_pX$ always has dimension equal to $\dim X$, even for $p \in \partial X$. It remains to check that the transition matrix between representations (2.5), for different coordinates, is $\mathcal{C}^\infty$. If $x', y'_1, \dots, y'_n$ are new coordinates then

$$\begin{aligned} x' &= xA(x,y) \text{ with } A(0,0) > 0 \text{ and} \\ y'_j &= Y_j(x,y) \text{ with } \det \frac{\partial Y_j}{\partial y_k}(0,0) \neq 0. \end{aligned}$$

Thus

$$\begin{aligned} x\frac{\partial}{\partial x} &= \big(1 + \frac{x}{A}\frac{\partial A}{\partial x}\big)x'\frac{\partial}{\partial x'} + \frac{x'}{A}\sum_{j=1}^n \frac{\partial Y_j}{\partial x}\frac{\partial}{\partial y'_j} \text{ and} \\ \frac{\partial}{\partial y_k} &= \frac{1}{A}\frac{\partial A}{\partial y_k}x'\frac{\partial}{\partial x'} + \sum_{j=1}^n \frac{\partial Y_j}{\partial y_k}\frac{\partial}{\partial y'_j}. \end{aligned} \tag{2.6}$$

This shows that the transition matrix is $\mathcal{C}^\infty$, so ${}^bTX$ is a well-defined vector bundle over X and (2.3) holds.

A more general version of this construction of ${}^bTX$, from TX, is described in Chapter 8 as the rescaling of vector bundles. For a discussion of ${}^bTX$ in case X is a manifold with corners see [66] and [63].

EXERCISE 2.6. Show that there is a natural vector bundle map from the cotangent bundle of X, T^*X, to the dual bundle to ${}^bTX$, denoted ${}^bT^*X$, and that this map is an isomorphism over the interior of X. Near any boundary point show that ${}^bT^*X$ has a coordinate basis which pulls back under this bundle map to

$$\frac{dx}{x}, dy_j,\ j = 1, \ldots, n. \tag{2.7}$$

2.3. Exact b-metrics.

Now, by b-geometry is meant the analogue of Riemannian geometry for the b-tangent bundle, ${}^bTX$. Thus a b-metric on a compact manifold with boundary is simply a metric on the fibres of ${}^bTX$. Over the interior of X,

$${}^bT_{X\setminus\partial X}X \simeq T_{X\setminus\partial X}X,$$

so a b-metric is a Riemann metric on the interior of X with special uniformity properties at the boundary. Indeed, using Exercise 2.6, in local coordinates at the boundary a b-metric can be written in the form

$$g = a_{00}(\frac{dx}{x})^2 + 2\sum_{j=1}^{n} a_{0j}\frac{dx}{x}dy^j + \sum_{j,k=1}^{n} a_{jk}dy^j dy^k, \tag{2.8}$$

where the coefficients are $\mathcal{C}^\infty$ and the form is positive definite:

$$a_{00}\lambda^2 + 2\sum_{j=1}^{n} a_{0j}\eta_j\lambda + \sum_{j,k=1}^{n} a_{jk}\eta_j\eta_k \geq \epsilon(|\lambda|^2 + |y|^2)\ \forall\ \lambda \in \mathbb{R}, \eta \in \mathbb{R}^n.$$

In practice it is convenient to further restrict the class of metrics. First note some of the intrinsic structure of ${}^bTX$. From (2.6) the element $x\partial/\partial x$ is, at a boundary point, completely invariant:

$$x\frac{\partial}{\partial x} \in T_pX \text{ is well-defined at } p \in \partial X. \tag{2.9}$$

Hence its span ${}^bN\partial X \subset {}^bT_{\partial X}X$ is a canonically trivial line subbundle.

EXERCISE 2.7. Show that there is a short exact sequence of vector bundles

$$0 \longrightarrow {}^bN\partial X \hookrightarrow {}^bT_{\partial X}X \longrightarrow T\partial X \longrightarrow 0 \tag{2.10}$$

where the projection is given by restriction to the boundary.

Now (2.9) shows the condition $(a_{00})_{\restriction\partial X} = 1$ on (2.8) to be meaningful independent of coordinates. This restriction will be strengthened further:

DEFINITION 2.8. An *exact b-metric* on a compact manifold with boundary is a b-metric such that for some boundary defining function $x \in \mathcal{C}^\infty(X)$

$$g = (\frac{dx}{x})^2 + g',\ g' \in \mathcal{C}^\infty(X; T^*X \otimes T^*X). \tag{2.11}$$

That is, for some choice of x, $a_{00} = 1 + O(x^2)$ and $a_{0j} = O(x)$ in (2.8).

EXERCISE 2.9. (An intrinsic characterization of exact b-metrics.) From one point of view (2.11) is rather forced since it just *demands* the exactness of g. It is possible to do a little better. First notice that one can just require that, for some boundary defining function x,

$$(a_{00})_{\restriction\partial X} = 1,\ (a_{0j})_{\restriction\partial X} = 0. \tag{2.12}$$

This means that there may be an additional singular term, $\gamma x(\frac{dx}{x})^2$ in (2.11). However setting $x' = x + \frac{1}{2}\gamma x^2$ eliminates this term, without creating any new singular terms. Thus it is only necessary to characterize (2.12). The first condition

$$g = (\frac{dx}{x})^2 \text{ on } {}^bN\partial X \tag{2.13}$$

has already been discussed.

Consider the subbundle which is the g-orthocomplement of ${}^bN\partial X$:

$$({}^bN\partial X)^\perp \subset {}^bT_{\partial X}X. \tag{2.14}$$

Going back to (2.4) observe that $\mathcal{C}^\infty(\partial X; {}^bT_{\partial X}X) = \mathcal{V}_b(X)/x\mathcal{V}_b(X)$. Moreover, as already noted, $\mathcal{V}_b(X)$ is a *Lie algebra* and furthermore $x\mathcal{V}_b(X) \subset \mathcal{V}_b(X)$ is an *ideal.* Thus the quotient, $\mathcal{C}^\infty(\partial X; {}^bT_{\partial X}X)$, is also a Lie algebra. From (2.14) the condition

$$\mathcal{C}^\infty(\partial X; ({}^bN\partial X)^\perp) \subset \mathcal{C}^\infty(\partial X; {}^bT_{\partial X}X) \text{ is a Lie subalgebra} \tag{2.15}$$

can be imposed. A metric satisfying (2.13), (2.15) might well be called a *closed* b-metric. Why? Check that these conditions do not *quite* ensure

(2.12), because such coordinates need only exist locally. Observe however that, given (2.15),

$$\begin{array}{c}({}^bN\partial X)^\perp \\ \downarrow \\ \partial X\end{array}$$

becomes a bundle with a flat connection. Show that g is *exact* if and only if the connection is a product connection, i.e. has trivial holonomy.

An exact b-metric fixes a section of $N^*\partial X$, namely dx, up to a global constant multiple on each component of ∂X by demanding that (2.11) hold. Indeed if x' is another such boundary defining function then $(dx'/x')^2 - (dx/x)^2$ must be a $\mathcal{C}^\infty$ quadratic 1-form. Writing $x' = \phi x$, with $\phi > 0$ and $\mathcal{C}^\infty$, this difference is $2d\phi dx/x\phi + (d\phi/\phi)^2$ so it follows that $d\phi = 0$ on ∂X. Thus ϕ restricts to ∂X to be locally constant and positive. Such trivializations are useful at various points below and the first coordinate in any coordinate system at the boundary will be taken to correspond to one.

2.4. Differential operators.

Next elementary differential geometry will be developed in this b-context, starting with forms and exterior differentiation. Let ${}^b\Lambda^k X = \Lambda^k\left({}^bT^*X\right)$ be the k^{th} exterior power of ${}^bT^*X$, i.e. the totally antisymmetric part of the k-fold tensor product. As always there is a canonical isomorphism

$$ {}^b\Lambda^k_{X\setminus\partial X} \simeq \Lambda^k_{X\setminus\partial X}. \tag{2.16}$$

The de Rham complex on any manifold (including a manifold with boundary) is

$$0 \to \mathcal{C}^\infty(X) \xrightarrow{d} \mathcal{C}^\infty(X;\Lambda^1) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{C}^\infty(X;\Lambda^N) \to 0, \tag{2.17}$$

where $N = \dim X$. The two main features to note are

$$d \in \mathrm{Diff}^1(X;\Lambda^k,\Lambda^{k+1}),\ d^2 = 0. \tag{2.18}$$

Recall what the first of these means, that d is a first order differential operator. The simplest definition is just to say that $P \in \mathrm{Diff}^k(X;V,W)$ if in any local trivializations of V and W over an open set P is given by a matrix of differential operators of order at most k. For the form bundles local trivializations are always induced from local coordinates and

$$\begin{gathered} u = \sum_{|\alpha|=k} u_\alpha dz^\alpha \Longrightarrow \\ du = \sum_{|\alpha|=k} du_\alpha \wedge dz^\alpha = \sum_{|\alpha|=k}\sum_{j=1}^N \frac{\partial u_\alpha}{\partial z^j} dz^j \wedge dz^\alpha. \end{gathered} \tag{2.19}$$

Thus d is a first order differential operator, as stated in (2.18).

The fact that the order of a differential operator is well defined depends ultimately on the fact that $\mathcal{V}(X) = \mathcal{C}^\infty(X; TX)$ is a Lie algebra. Thus for $V = W = \mathbb{C}$ a differential operator of order k acting on functions is just the sum of up to k-fold products of vector fields:

$$\operatorname{Diff}^k(X) = \operatorname*{span}_{0\le j\le k} \mathcal{V}(X)^j, \ \mathcal{V}(X)^0 = \mathcal{C}^\infty(X).$$

Using the corresponding Lie algebra in the b setting, $\mathcal{V}_b(X)$, the same definition leads to b-differential operators.

DEFINITION 2.10. For any manifold with boundary, the space $\operatorname{Diff}^k_b(X)$ of b-differential operators of order k consists of those linear maps $P : \mathcal{C}^\infty(X) \longrightarrow \mathcal{C}^\infty(X)$ given by a finite sum of up to k-fold products of elements of $\mathcal{V}_b(X)$ (and $\mathcal{C}^\infty(X)$)

$$P = \sum_{l, k'(l)\le k} a_{(l)} \cdot V_{1,l} \dots V_{k',l}, \ V_{j,l} \in \mathcal{V}_b(X), \ a_{(l)} \in \mathcal{C}^\infty(X). \tag{2.20}$$

If V, W are vector bundles then $\operatorname{Diff}^k_b(X; V, W)$ consists of the operators $P : \mathcal{C}^\infty(X; V) \longrightarrow \mathcal{C}^\infty(X; W)$ which are local (i.e. decrease supports) and given in any common trivialization of V and W over an open subset of X by a b-differential operator.

The formal point is that the de Rham complex (2.17) over $X \setminus \partial X$ lifts to the 'b' version of the de Rham complex:

$$0 \to \mathcal{C}^\infty(X) \xrightarrow{{}^b d} \mathcal{C}^\infty(X; {}^b\Lambda^1) \xrightarrow{{}^b d} \dots \xrightarrow{{}^b d} \mathcal{C}^\infty(X; {}^b\Lambda^{n+1}) \to 0, \tag{2.21}$$

where $\dim X = n+1$ and ${}^b d$ is just the lift of d to an operator on the b-form bundles using (2.16). In fact ${}^b d \in \operatorname{Diff}^1_b(X; {}^b\Lambda^k, {}^b\Lambda^{k+1})$ for all k. To see this, start with the formula (2.19) and the basis (2.7) and observe that near the boundary

$$u \in \mathcal{C}^\infty(X; {}^b\Lambda^k) \Longrightarrow u = \sum_{|\alpha|=k} u_\alpha dy^\alpha + \sum_{|\alpha|=k-1} u'_\alpha \frac{dx}{x} \wedge dy^\alpha,$$

with the coefficients $\mathcal{C}^\infty$, and then

$$\begin{aligned} {}^b du = \sum_{|\alpha|=k} (x\frac{\partial u_\alpha}{\partial x}) \frac{dx}{x} \wedge dy^\alpha + \sum_{|\alpha|=k} \sum_{j=1}^n \frac{\partial u_\alpha}{\partial y_j} dy_j \wedge dy^\alpha \\ - \sum_{|\alpha|=k-1} \sum_{j=1}^n \frac{\partial u'_\alpha}{\partial y_j} \frac{dx}{x} \wedge dy_j \wedge dy^\alpha. \end{aligned}$$

For reasons of sanity-preservation it is better to set ${}^b d = d$ again.

An element of $\mathcal{V}_b(X)$ can be viewed in several ways. In particular it can be viewed as an element of $\mathrm{Diff}^1_b(X)$. It can also be considered as an element of $\mathcal{C}^\infty(X; {}^bTX)$ and hence as a linear function on the fibres of the dual bundle, i.e. as an element of $P^{[k]}({}^bT^*X)$, the space of $\mathcal{C}^\infty$ functions on ${}^bT^*X$ which are homogeneous polynomials of degree k on the fibres, for $k=1$. Let ${}^b\sigma_1(V) \in P^{[1]}({}^bT^*X)$ denote the image of $iV \in \mathcal{V}_b(X)$ in this sense. The factor of 'i' is inserted so that D_{y_j} and xD_x have real symbols, where $D_{y_j} = -i\partial/\partial y_j$, $xD_x = -ix\partial/\partial x$ corresponding to normalization of the Fourier and Mellin transforms. Since $\mathcal{V}_b(X)$ is a Lie algebra the map defined by taking the leading terms in (2.20):

(2.22)
$$\mathrm{Diff}^k_b(X) \ni P \longmapsto {}^b\sigma_k(P) = \sum_{l, k'(l)=k} a_{(l)}\, {}^b\sigma_1(V_{1,l}) \cdots {}^b\sigma_1(V_{k,l}) \in P^{[k]}({}^bT^*X)$$

is defined independently of the choice of presentation (2.20).

In the interior of X where ${}^bTX$ is identified with TX and ${}^bT^*X$ with T^*X this gives the symbol in the usual sense. In adapted local coordinates at the boundary

$$P \in \mathrm{Diff}^k_b(X) \Longleftrightarrow P = \sum_{j+|\alpha|\le k} p_{j,\alpha}(x,y)(xD_x)^j D_y^\alpha \tag{2.23}$$

and then the symbol becomes, in terms of the basis (2.7),

$${}^b\sigma_k(P) = \sum_{j+|\alpha|=k} p_{j,\alpha}(x,y)\lambda^j\eta^\alpha. \tag{2.24}$$

Here λ and η are the linear functions on ${}^bT^*X$ defined by the basis, i.e. a general element of ${}^bT^*X$ is written

$$\lambda\frac{dx}{x} + \sum_{i=1}^n \eta_i dy_i.$$

An operator $P \in \mathrm{Diff}^k_b(X)$ is said to be elliptic (really b-elliptic) if ${}^b\sigma_k(P) \neq 0$ on ${}^bT^*X \setminus 0$.

For an operator $P \in \mathrm{Diff}^k_b(X; V, W)$, between vector bundles, the coefficients in (2.24) become local homomorphisms from V to W. Thus the symbol is a well-defined element of the space of homomorphisms from the lift of V to ${}^bT^*X$ to the lift of W and is a homogeneous polynomial of degree k on the fibres:

$${}^b\sigma_k(P) \in P^{[k]}({}^bT^*X; V, W),\ P \in \mathrm{Diff}^k_b(X; V, W). \tag{2.25}$$

Such an operator is elliptic if ${}^b\sigma_k(P)$ is invertible on ${}^bT^*X \setminus 0$.

Exercise 2.11. Show that the symbol of $d \in \mathrm{Diff}^1_b(X; {}^b\Lambda^k, {}^b\Lambda^{k+1})$ at $\zeta \in {}^bT^*X$ is

$$ {}^b\sigma_1(d)(\zeta) = i\zeta \wedge . \tag{2.26}$$

2.5. Levi-Civita Connection.

Having dealt with the most elementary parts of differential geometry, consider next the fundamental notions in Riemannian geometry, especially the Levi-Civita connection and Riemann curvature tensor. As above this will be developed with an eye to the generalization to the case of a b-metric. If X is a $\mathcal{C}^\infty$ manifold, then a Riemann metric on X is just a positive-definite fibre metric on the tangent bundle. Since it is non-degenerate, such a metric, g, fixes an isomorphism

$$TX \xrightarrow{G} T^*X,\ G(v)(w) = g_p(v,w)\ \forall\ v,w \in T_pX\ \forall\ p \in X.$$

This means that the metric can be transferred to the dual bundle, T^*X.

An obvious question to ask is the extent to which the metric can be simplified by choice of local coordinates. The metric g is said to be *flat* near p if it reduces to the Euclidean metric in some local coordinates. How close to flat is a metric? A metric is always flat up to second-order terms in Taylor series around any point. This is the origin of the Levi-Civita connection.

Certainly for any point $p \in X$ there are local coordinates based at p (meaning in which p is mapped to the origin), $z^1, z^2, \dots, z^N$, such that

$$g_p = (dz^1)^2 + \cdots + (dz^N)^2 \text{ on } T_pX. \tag{2.27}$$

In fact any basis of T^*_pX is given by the differentials at p of some coordinate system, so one can just as well look at the space of orthonormal bases of T^*_pX with respect to g_p :

$$F_p = \left\{ \phi = (\phi^1, \dots, \phi^N); \phi^j \in T^*_pX \text{ and } g_p = \sum_{j=1}^N (\phi^j)^2 \right\}.$$

Notice that if $\phi = (\phi^1, \dots, \phi^N)$ is one element of F_p and $O \in \mathrm{O}(N)$ is an orthogonal matrix, i.e. a real $N \times N$ matrix with inverse O^t, then

$$\psi = (\psi^1, \dots, \psi^N) \in F_p,\ \psi^j = \sum_{k=1}^N O_{kj}\phi^k. \tag{2.28}$$

Every element of F_p arises in this way from a fixed element and O can be recovered from ψ and ϕ. The action of $\mathrm{O}(N)$ on F_p given by (2.28)

$$A : F_p \times \mathrm{O}(N) \longrightarrow F_p,\ (\phi, O) \longmapsto \phi O = \psi \tag{2.29}$$

is simple and transitive. The action is on the right:

$$A(A(\phi, O), O') = A(\phi, OO'),\ \text{ i.e. } (\phi O)O' = \phi(OO').$$

As p varies locally in X the metric is reduced to the Euclidean metric by a smoothly varying basis of T^*X. Thus the space

$$F = \bigsqcup_{p \in X} F_p \stackrel{\pi}{\longrightarrow} X$$

is itself a $\mathcal{C}^\infty$ manifold such that the basis is a smooth section and the $\mathrm{O}(N)$ action, given by (2.29) on each fibre, is $\mathcal{C}^\infty$. In fact F is just the orthonormal coframe bundle of the Riemann manifold X with metric g; it is a principal $\mathrm{O}(N)$-bundle. The map (2.29) will be written as

$$A_O : F \longrightarrow F \quad \forall\ O \in \mathrm{O}(N).$$

It is also useful to consider the closely related map $L_\phi \colon F_p \longleftrightarrow \mathrm{O}(N)$, fixed for each $p \in X$ and each $\phi \in F_p$ by

$$L_\phi(\psi) = O \text{ if } A_O \phi = \psi. \tag{2.30}$$

Going back to the question of simplifying the metric, choose any coordinates based at p in which (2.27) holds. Thus, near p, the metric is of the form

$$g = (dz^1)^2 + \cdots + (dz^N)^2 + \sum_{i,j,k=1}^{N} g_{ij,k} dz^i dz^j z^k + O(|z|^2), \tag{2.31}$$

where $g_{ij,k} = \partial g_{ij}/\partial z^k$ at p is a constant matrix satisfying

$$g_{ij,k} = g_{ji,k} \quad \forall\ i, j, k = 1, \ldots, N. \tag{2.32}$$

The coordinates can be modified, without changing the basis at p, by adding quadratic terms:

$$w^j = z^j + \frac{1}{2} \sum_{l,k=1}^{N} \gamma^j_{lk} z^l z^k,\ j = 1, \ldots, N, \tag{2.33}$$

where the coefficients are arbitrary except for the symmetry condition

$$\gamma^j_{lk} = \gamma^j_{kl} \quad \forall\ j, k, l. \tag{2.34}$$

The inverse of this transformation is, near p, of the form:

$$z^j = w^j - \frac{1}{2} \sum_{lk=1}^{N} \gamma^j_{lk} w^l w^k + O(|w|^3).$$

Inserting this into (2.31) the metric becomes

$$g = (dw^1)^2 + \cdots + (dw^N)^2 + O(|w|^2) \tag{2.35}$$

if and only if the linear terms in z cancel, i.e.

$$\sum_{i,j,k=1}^{N} g_{ij,k} dz^i dz^j z^k = \sum_{j,l,k=1}^{N} \gamma^j_{lk} (dz^j z^l dz^k + dz^j dz^l z^k).$$

This is a symmetric form in dz, so, using (2.34), the condition can be written in terms of the coefficients as

$$\gamma^j_{lk} + \gamma^l_{jk} = g_{jl,k} \quad \forall\ j, k, l. \tag{2.36}$$

Lemma 2.12. *The linear map $\gamma^j_{lk} \longmapsto g_{ij,k}$ given by* (2.36) *is an isomorphism of matrices subject to* (2.34) *onto matrices subject to* (2.32).

Proof: The inverse of (2.36) can be computed explicitly as

$$\gamma^j_{lk} = \frac{1}{2} \left\{ g_{jk,l} + g_{jl,k} - g_{lk,j} \right\}. \tag{2.37}$$

This means that once the basis $\phi \in F_p$ is chosen there is a *unique* way of choosing the coordinates, up to cubic terms, so that $\phi = (dw^1, \ldots, dw^n)$ at p and (2.35) holds. This information is elegantly encoded in the concept of a *connection* on F. If $w^1, \ldots, w^N$ are the coordinates constructed so that (2.35) holds then the coordinate basis

$$dw^1, \ldots, dw^N$$

can be orthonormalized, near p, by the Gram-Schmidt procedure and this only changes it by quadratic terms. Thus there is a local orthonormal coframe $\phi^1, \ldots, \phi^N$ of the form

$$\phi^j = dw^j + \sum_{k=1}^{N} a_{jk} dw^k, \ a_{jk} = O(|w|^2). \tag{2.38}$$

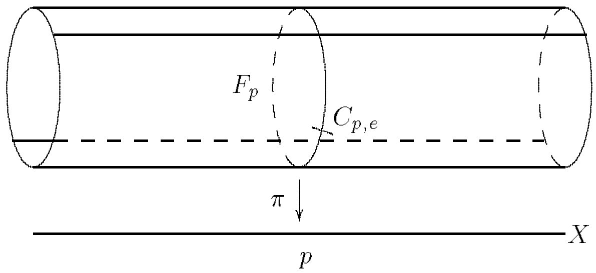

Figure 3. The orthonormal frame bundle and connection.

If the $\phi^j(p)$ are a given orthonormal basis of T_p^*X then the ϕ^j are determined by (2.38) up to quadratic terms at p. This defines an N-plane in the tangent space to F at each point (p, ϕ) :

$$C_{p,\phi} \subset T_{(p,\phi)}F, \;\; \phi \in F_p. \tag{2.39}$$

The uniqueness of the planes $C_{p,\phi}$ shows that the differential of the $\mathrm{O}(N)$-action intertwines the spaces

$$(A_O)_*(C_{p,\phi}) = C_{p,\phi O} \quad \forall \; O \in \mathrm{O}(N) \tag{2.40}$$

and clearly

$$C_{p,\phi} \oplus T_\phi F_p = T_{(p,\phi)}F \quad \forall \; p, \phi \tag{2.41}$$

since $C_{p,\phi}$ is the tangent plane to a local section of F. These are precisely the conditions defining a connection on F. That is:

DEFINITION 2.13. A connection on a principal $\mathrm{O}(N)$-bundle is a smooth N-dimensional distribution, i.e. assignment of an N-plane $C_{p,\phi}$ at each point (p, ϕ) of the bundle, satisfying (2.40) and (2.41).

Thus the discussion above just leads to the Levi-Civita connection:

LEMMA 2.14. *The prescription* (2.39) *fixes a connection (the Levi-Civita connection) on the orthonormal coframe bundle of any Riemann manifold.*

An orthonormal coframe ϕ^l is said to be covariant constant at p if the tangent space at $(p, \phi(p))$ to the section of F defined by it is $C_{p,\phi(p)}$.

2.6. Riemann curvature tensor.

Cartan's reformulation of this notion of a connection is based on the observation that, for each $\phi \in F_p$, the identification L_ϕ in (2.30) maps ϕ to the identity in $\mathrm{O}(N)$, so if the tangent space to $\mathrm{O}(N)$ at Id is taken as the Lie algebra, $\mathfrak{so}(N)$, then

$$(L_\phi)_* : T_\phi F_p \longleftrightarrow \mathfrak{so}(N).$$

Therefore there is a well-defined 1-form with values in $\mathfrak{so}(n)$

$$\omega \in \mathcal{C}^\infty(F; \Lambda^1 F \otimes \mathfrak{so}(N)), \tag{2.42}$$

which is to say N^2 1-forms

$$\omega_{ij} \in \mathcal{C}^\infty(F; \Lambda^1 F) \text{ s.t. } \omega_{ij} + \omega_{ji} = 0 \quad \forall\ i, j = 1, \dots, N,$$

fixed by the condition that at each point of F

$$\omega(v) = (L_\phi)_* v,\ \ v \in T_\phi F_p,\ \ \phi \in F_p \tag{2.43}$$

and

$$\omega_{\restriction C_{p,\phi}} = 0. \tag{2.44}$$

That this fixes the connection form, ω, uniquely follows from (2.41) and then from (2.40) if follows that

$$(A_O)^* \omega_{ij} = \sum_{p,q} O_{ip} O_{jq} \omega_{pq} \quad \forall\ O \in \mathrm{O}(N) \tag{2.45}$$

gives the transformation law under the $\mathrm{O}(N)$-action. Conversely if a 1-form (2.42) satisfying (2.43) and (2.45) is given and the planes $C_{p,\phi} \subset T_{p,\phi} F$ are defined to be the null spaces of ω, so (2.44) holds, then the $C_{p,\phi}$ define a connection on F.

There is another way to look at the connection forms ω_{ij}, more in keeping with the method of moving frames. There are N natural evaluation maps $\Xi_i \colon F \longrightarrow T^*X$, just sending an orthonormal coframe to its ith element. The cotangent bundle carries a *tautological* form, α, given at any point $(p, \beta) \in T^*X$ by the pull back of β under the projection from T^*X to X. Thus F has N such tautological forms, $\alpha_i = \Xi_i^* \alpha$, $i = 1, \dots, N$. These forms vanish when restricted to any fibre but are everywhere independent. The *first structure equation*[3] of the connection is:

$$d\alpha_i + \sum_{p=1}^{N} \omega_{ip} \wedge \alpha_p = 0. \tag{2.46}$$

Notice that this actually determines the ω_{ij}.

To prove (2.46) at a general point $(p, \phi) \in F$ consider an orthonormal frame as in (2.38), near p, taking the value ϕ at p. This gives an identification over a neighbourhood U of p :

$$\begin{aligned} F_{\restriction U} &\equiv U \times \mathrm{O}(N) \\ (p', \psi) \longleftrightarrow (p', O),\ \psi^j &= \sum_k O_{kj} \phi^k \end{aligned} \tag{2.47}$$

as in (2.28). Then the tautological forms are just $\alpha_i = \sum_l O_{li}\phi^l$. Moreover, $d\phi^l$ vanishes at p so $d\alpha_i = \sum_l dO_{li} \wedge \phi^l$ at $p \times \mathrm{O}(N)$. Inverting (2.47) and noting that the covariant constancy of the basis means that

$$\omega_{ij} = \sum_q O_{qi} dO_{qj} \text{ at } p \times \mathrm{O}(N) = F_p;$$

it follows that

$$d\alpha_i = \sum_l dO_{li} \wedge \sum_q O_{lq}\alpha_q = \sum_q \omega_{qi} \wedge \alpha_q.$$

This gives (2.46) at (p, ϕ) and hence in general.

The *second structure equation* for the connection is given by computing the exterior differential of the connection form.

PROPOSITION 2.15. *The Levi-Civita connection form, ω, defined by* (2.43) *and* (2.44), *satisfies*

$$d\omega_{ij} + \sum_q \omega_{iq} \wedge \omega_{qj} = Q_{ij} \tag{2.48}$$

[3] The convention used here for the exterior product is fixed by demanding

$$\xi_1 \wedge \cdots \wedge \xi_N(v_1, \ldots, v_N) = \frac{1}{N!} \det(\xi_j(v_k)) \quad \forall\, \xi_j \in T_p^*X,\ v_j \in T_pX.$$

where $Q_{ij} \in \mathcal{C}^\infty(F; \Lambda^2 F)$ annihilates $T_\phi F_p$ at each point, i.e.

$$Q(v, w) = 0 \text{ if } v \in T_{(p,\phi)}F,\ w \in T_\phi F_p. \tag{2.49}$$

PROOF: Any tangent vector $v \in T_\phi F$ can be extended to a vector field, V, near F_p. If $v \in T_\phi F_p$ the vector field V can be chosen to be vertical, i.e. tangent to the leaves of F, near p and to have

$$(L_\psi)_* V = v' = (L_\phi)_* v \ \forall\ \psi \in F_\phi,$$

so constant in $\mathfrak{so}(N)$. This means that V generates a 1-parameter subgroup of the $\mathrm{O}(N)$-action.

Suppose first that $v, w \in T_\phi F_p$ are both vertical. Then, by Cartan's formula for the exterior derivative,

$$d\omega(V, W) = \frac{1}{2}\left[V\omega(W) - W\omega(V) - \omega([V, W])\right]. \tag{2.50}$$

Taking V, W to be generators of 1-parameter subgroups of the $\mathrm{O}(N)$-action as above makes $\omega(V)$ and $\omega(W)$ constant, so the first two terms on the right in (2.50) vanish. Evaluated at (p, ϕ), using (2.43), the third term in (2.50) becomes

$$\begin{aligned}\omega([V, W]) =& (L_\phi)_*([V, W])\\ =& [(L_\phi)_* V, (L_\phi)_* W]\\ =& \omega(V)\omega(W) - \omega(W)\omega(V)\end{aligned}$$

since the commutator in $\mathfrak{so}(N)$ is just matrix commutation. This gives (2.49) when both vectors are vertical.

Suppose next that $v \in C_{p,\phi}$. Then, using (2.40), the extension can be chosen to be $\mathrm{O}(N)$-invariant and must take values in $C_{p,\psi}$ for all $\psi \in F_p$. Thus $\omega(V) = 0$ on F_p. With W a generator of the $\mathrm{O}(N)$-action the first two terms in (2.50) again vanish. Now $[W, V]$ is the Lie derivative of V along the 1-parameter group generated by W, so vanishes on F_p and hence the third term in (2.50) also vanishes at ϕ. This completes the proof of (2.49).

From (2.49) the form Q at $(p, \phi) \in F$ is determined by its restriction to $C_{p,\phi}$. By (2.41) projection to X gives an isomorphism of $T_p X$ and $C_{p,\phi}$, so one can define

$$\begin{gathered}Q_{ij}^{p,\phi} \in \Lambda_p^2 X \text{ by } Q_{ij}^{p,\phi}(v, w) = Q_{ij}(v', w'),\\ v', w' \in C_{p,\phi},\ \pi_*(v') = v, \pi_*(w') = w.\end{gathered}$$

Then consider the 4-cotensor at p :

$$\begin{gathered}R(p; v, w, \tilde{v}, \tilde{w}) = \sum_{i,j=1}^{n} Q_{ij}^{p,\phi}(v, w)\phi^i(\tilde{v})\phi^j(\tilde{w}),\\ v, w, \tilde{v}, \tilde{w} \in T_p X,\ \phi = (\phi^1, \dots, \phi^N) \in F_p.\end{gathered} \tag{2.51}$$

The notation here indicates that

$$R(p) \in \Lambda_p^2 X \otimes \Lambda_p^2 X$$

is independent of the choice of $\phi \in F_p$ involved in (2.51). This indeed is a consequence of the transformation law for Q in (2.48) which follows from (2.45):

$$(A_O)^* Q_{ij} = \sum_{p,q} O_{ip} O_{jq} Q_{pq} \tag{2.52}$$

and the transformation law (2.28) for the ϕ^i. Thus

$$R \in \mathcal{C}^\infty(X; \Lambda^2 X \otimes \Lambda^2 X)$$

is well defined by (2.51). It is the *Riemann curvature tensor.* Notice that, as a consequence of Proposition 2.15, R also determines Q.

THEOREM 2.16. *If the Riemann curvature tensor vanishes near a point p then the metric is flat nearby in the sense that there are local coordinates based at p in terms of which*

$$g = (dz^1)^2 + \cdots + (dz^N)^2 \text{ near } p.$$

PROOF (BRIEF): By Proposition 2.15 and the remarks above the vanishing of R is equivalent to the vanishing of Q in a neighbourhood of F_p in F and hence to

$$d\omega = -\omega \wedge \omega \text{ near } F_p.$$

This in turn means that the subspaces $C_{p,\phi} \subset T_{(p,\phi)}F$ form a foliation near F_p, i.e. the space of vector fields tangent to them at each point is closed under commutation. So, by Frobenius' theorem, through each point $\phi \in F_p$ there is, at least locally, a smooth submanifold, X_ϕ, such that $T_{(p',\psi)}X_e = C_{p',\psi}$ for each $(p',\psi) \in X_\phi$. The transversality condition (2.41) means that $\pi : X_\phi \longrightarrow X$ is a local diffeomorphism, so X_ϕ is a section of F passing through (p,ϕ), i.e. an orthonormal basis $\phi^i(p')$ for g at each point p' near p. The tangency of $C_{p',\psi}$ to X_ϕ means that each $\phi^i(p')$ is actually a closed 1-form, so locally exact and this gives the coordinates in which the metric is flat.

So this is the elementary theory of the Levi-Civita connection and Riemann curvature of a Riemann manifold. Before considering the changes needed to handle b-metrics, recall the identification of the tensor bundles as *associated bundles* to the coframe bundle and the extension of the connection to these bundles in the guise of *covariant differentiation.*

2.7. Associated bundles.

Suppose that E is a vector space with an $\mathrm{O}(N)$-action, $L\colon\ \mathrm{O}(N) \ni O \longmapsto L_O \in \mathrm{Gl}(E)$. For example consider the standard action on $\mathbb{R}^N$:

$$L_O(x_1, \dots, x_N) = (y_1, \dots, y_N),\ y_j = \sum_{k=1}^{N} O_{jk} x_k. \tag{2.53}$$

Given any such linear action consider the space $F \times E$ with the action:

$$F \times E \in (p, \phi, u) \longrightarrow (p, A_O\phi, L_O^{-1}u) \in F \times E. \tag{2.54}$$

The quotient is a vector bundle over X, which can be denoted temporarily as E_L :

$$\begin{aligned}(E_L)_p = \big\{[(p,\phi,u)];(p,\psi,u') \in [(p,\phi,u)] \text{ iff } \exists\ O \in \mathrm{O}(N)\\ \text{with } \psi = A_O\phi = \phi O,\ u' = L_O^{-1}u\big\}.\end{aligned}$$

The linear structure on $(E_L)_p$ comes from the linearity of the action on E, the $\mathcal{C}^\infty$ structure projects from $F \times E$.

Any such associated bundle has a connection induced from F. With $C_{p,\phi}$ the N-planes defined above consider

$$C^L_{p,\phi,u} = C_{p,\phi} \times E \subset T_{(p,\phi,u)}(F \times E)\ \forall\ (p,\phi,u) \in F \times E,$$

where the identification of the tangent space of a vector space with the vector space itself is used. Now from (2.40) and the fact that L_O acts on, i.e. preserves, E the differential of the action (2.54) maps $C^L_{p,\phi,u}$ onto $C^L_{p,\phi O,L_O^{-1}u}$. Thus the spaces $C^L_{p,\phi,u}$ project to well-defined N-planes in $T_{(p,v)}E_L$, which can again be denoted

$$C_{p,v} \subset T_{(p,v)}E_L,\ v \in (E_L)_p,\ p \in X.$$

The transversality condition (2.41) certainly persists so

$$C_{p,v} \oplus T_v(E_L)_p = T_{(p,v)}E_L \quad \forall\ (p,v) \in E_L. \tag{2.55}$$

EXERCISE 2.17. Check carefully that if the representation of $\mathrm{O}(N)$ is taken to be (2.53) then the associated bundle is canonically isomorphic to T^*X where the isomorphism is given by mapping (p,ϕ,u) to

$$\sum_j u_j \phi^j \in T^*_pX.$$

2.8. Covariant differentiation.

It is more usual to rewrite the information (2.55) in terms of a differential operator. A section, $\mu \in \mathcal{C}^\infty(X; E_L)$, is said to be covariant constant at p if its graph

$$\mathrm{Gr}(\mu) = \{(p', \mu(p')) \in E_L; p' \in X\} \text{ has tangent plane } C_{p,\mu(p)} \text{ at } (p, \mu(p)).$$

More generally any section, at any point, can be compared to a covariant constant section. Since

$$T_{(p,\mu(p))}\,\mathrm{Gr}(\mu) \oplus T_{\mu(p)}(E_L)_p = T_{(p,\mu(p))}E_L,$$

there is a well-defined projection $\nabla\mu(p) \colon C_{p,\mu(p)} \longrightarrow T_{\mu(p)}(E_L)_p$. Since the projection to the base identifies $C_{p,\mu(p)}$ with T_pX and the tangent space to the fibre can be identified with the fibre this means that $\nabla\mu(p)$ is a linear map $T_pX \longrightarrow (E_L)_p$ fixed by μ. Equivalently it can be thought of it as an element

$$\nabla\mu(p) \in T_p^*X \otimes (E_L)_p.$$

Clearly this construction varies smoothly with p so defines the covariant derivative of any smooth section of any associated bundle:

$$\nabla : \mathcal{C}^\infty(X; E_L) \longrightarrow \mathcal{C}^\infty(X; T^*X \otimes E_L). \tag{2.56}$$

Lemma 2.18. *The map* (2.56) *is a first order linear differential operator.*

Proof: The tangent space to the section only depends on the first derivatives of the section in local coordinates, so certainly $\nabla\mu(p)$ only depends on the first derivatives, and the value, of μ at p. To see that the dependence is linear observe that, by definition, if ϕ is any section of F near p then a local section μ, of E_L, determines and is determined by a map f from a neighbourhood of p in X into E by $\mu(p') \longleftrightarrow [(\phi(p'), f(p'))]$. If ϕ is covariant-constant at p, i.e. has graph tangent to $C_{p,\phi(p)}$, then the covariant derivative of μ is just

$$\nabla\mu(p) = df(p) \in T_p^*X \otimes (E_L)_p, \tag{2.57}$$

where $E \ni e \longrightarrow [(\phi(p), e)]$ identifies E and $(E_L)_p$. The linearity is then clear from (2.57).

Directly from the definition, the zero section of the vector bundle is covariant constant, which is to say that at the zero section $C_{p,0} = T_pX$ is the tangent plane to the zero section. Note that the tangent space to the vector bundle has a natural decomposition $T_{p,0}E = T_pX \oplus E_p$. If μ is any smooth section and $f \in \mathcal{C}^\infty(X)$ vanishes at p then the graph of the section $f\mu$ has tangent space $\{df(p) = 0\} \oplus \operatorname{span}(\mu)$ at p. It follows that the covariant derivative of such a section, at p, is just $df(p) \otimes \mu(p)$. More generally

$$\nabla(f\mu) = df \otimes \mu + f\nabla\mu \quad \forall\ f \in \mathcal{C}^\infty(X),\ \mu \in \mathcal{C}^\infty(X; E). \tag{2.58}$$

Indeed this follows, at a general point p, by writing $f\mu = f(p)\mu + (f - f(p))\mu$.

DEFINITION 2.19. A connection on a vector bundle, E, is a first order linear differential operator

$$\nabla\colon \mathcal{C}^\infty(X; E) \longrightarrow \mathcal{C}^\infty(X; T^*X \otimes E)$$

satisfying (2.58).

The Levi-Civita connection on T^*X is certainly a connection (an affine connection) in this sense.

EXERCISE 2.20. Show that for any connection on a vector bundle over X and each $e \in E_p$, $p \in X$, there is a section $\mu \in \mathcal{C}^\infty(X; E)$ with $\mu(p) = e$ which is covariant constant at p, i.e. $\nabla\mu(p) = 0$. [Hint: choose any section through e at p then add to it $\sum_i f_i\mu_i$, where the μ_i are a local basis of sections and the $f_i \in \mathcal{I}_p$. Use (2.58) to choose the f_i so that the covariant derivative vanishes at p.] Use this to show that a connection is equivalent to a $\mathcal{C}^\infty$ distribution of tangent planes $C_{(p,e)} \subset T_{(p,e)}E$ which project isomorphically onto T_pX and satisfy a linearity condition which can be written

$$C_{(p,e)} + C_{(p,e')} = C_{(p,e+e')}\ \forall\ e, e' \in E_p,\ p \in X.$$

Thus if E has a connection and $p \in X$ one can always find a local basis of sections of E, e_j, which are covariant constant at p. Any other such basis is necessarily of the form

$$\tilde{e}_j = \sum_k a_{jk}e_k + \sum_{jk} f_{jk}e_k,\ a_{jk} \text{ constant and } f_{jk} \in \mathcal{I}_p^2. \tag{2.59}$$

If ∇' and ∇'' are both connections on E and $\phi \in \mathcal{C}^\infty(X)$ then $\nabla = \phi\nabla' + (1 - \phi)\nabla''$ is also a connection. Of course it is only necessary for ∇' to be defined in a neighbourhood of the support of ϕ and similarly for ∇''. This allows one to show that any vector bundle has a connection. For a trivial bundle, $E = X \times \mathbb{R}^n$, a connection is defined by d acting on the coefficients. Since any bundle is locally trivial, this means that a connection exists in an open neighbourhood of each point. If O_j is a covering of X by such neighbourhoods with ∇_j a connection on E over O_j and ϕ_j is a partition of unity subordinate to the cover then $\nabla = \sum_j \phi_j \nabla_j$ is a connection on E.

The covariant derivative can always be evaluated on a vector field (in the factor of T^*X in $T^*X \otimes E$). Thus if $V \in \mathcal{V}(X)$

$$\nabla_V : \mathcal{C}^\infty(X; E) \longrightarrow \mathcal{C}^\infty(X; E),\ \nabla_V \mu = \nabla\mu(V, \cdot)\ \forall\ \mu \in \mathcal{C}^\infty(X; E)$$

is always well defined.

LEMMA 2.21. *Every $\mathcal{C}^\infty$ bundle has a connection. If the bundle has a fibre metric then the connection can be chosen to be orthogonal in the sense that*

$$\langle \nabla_V \mu, \mu' \rangle + \langle \mu, \nabla_V \mu' \rangle = d\langle \mu, \mu' \rangle \quad \forall\ \mu, \mu' \in \mathcal{C}^\infty(X; E) \tag{2.60}$$

for all $\mathcal{C}^\infty$ vector fields V.

PROOF: Using the same superposition argument it suffices to construct an orthogonal connection locally, assuming the bundle to be trivial. Then one can simply demand that some orthonormal basis be covariant constant.

If E has a connection then the dual bundle E' has a natural connection given by demanding that a section e' of E' be covariant constant at p if and only if its pairing with any section of E which is covariant constant at p has vanishing differential at p. That is, in terms of the pairing

$$d\langle e, e' \rangle = \langle \nabla e', e \rangle + \langle e', \nabla e \rangle. \tag{2.61}$$

Indeed if e_j is a covariant constant basis at p and e'_j is the dual basis of E' at p then e'_j must be covariant constant. This fixes the connection on E' and it is independent of the choice of basis since another choice, (2.59), clearly gives the same covariant constant sections at p.

Similarly the tensor product of two bundles E and F with connections has a unique connection for which $e \otimes f$ is covariant constant at p if e and f are covariant constant at p as sections of E and F. It is determined by the identity

$$\nabla(e \otimes f) = (\nabla e) \otimes f + e \otimes (\nabla f). \tag{2.62}$$

EXERCISE 2.22. Prove these assertions carefully, in particular showing smoothness of the induced connections.

EXERCISE 2.23. Show that if E and F are bundles associated to a principal $\mathrm{O}(N)$-bundle then $E \otimes F$ is also an associated bundle, corresponding to the tensor product representation. Check that the tensor product of the connections induced on E and F, as in (2.62), is the induced connection on $E \otimes F$.

The Levi-Civita connection therefore defines covariant differentiation acting on tensors on X. Actually this connection arises in two ways, directly because the tensor bundles are associated to the orthonormal coframe bundle and also through the tensor product construction from the connections on the tangent and cotangent bundles; by Exercise 2.23 these are the same. Various properties of the Levi-Civita connection are important later. First since g, the metric, can be considered as a section of $T^*X \otimes T^*X$ (a symmetric one to be sure) its covariant derivative is defined. In fact $\nabla g \equiv 0$. Indeed this follows from (2.57) and the original definition of the connection, since if g is represented in terms of an orthonormal basis which is covariant constant at p it is given, by definition, by the Euclidean form up to second order.

EXERCISE 2.24. For any form bundle, antisymmetrization gives a projection $A_k : T^*X \otimes \Lambda^k X \longrightarrow \Lambda^{k+1} X$. Show that composition with covariant differentiation gives, for the Levi-Civita connection,

$$A_{k+1} \circ \nabla = d : \mathcal{C}^\infty(X; \Lambda^k) \longrightarrow \mathcal{C}^\infty(X; \Lambda^{k+1})$$

for every k.

EXERCISE 2.25. Using Exercise 2.24 make sense of, and prove, the following formula for any (local) orthonormal coframe ϕ^i with dual frame v_i :

$$d\mu = \sum_{j=1}^{n} \phi^i \wedge \nabla_{v_i} \mu, \tag{2.63}$$

where μ is a $\mathcal{C}^\infty$ k-form.

2.9. Christoffel symbols.

The construction above can be reinterpreted in terms of the full coframe bundle $\widetilde{F}$. Thus

$$\widetilde{F}_p = \left\{ \phi = \{\phi^1, \dots, \phi^N\} \text{ is a basis of } T_p^* X \right\}.$$

The N-plane $C_{p,\phi}$ can be defined at all points of $\widetilde{F}$, using (2.40), i.e. a general frame is covariant constant at p if and only if it is of the form

$\sum_j h^i_j \phi^j$, where ϕ^j is an orthonormal frame which is covariant constant at p and h^i_j is an invertible matrix with $dh^j_i = 0$ at p. In a general coordinate system, not assumed to give an orthonormal basis for T^*_pX, the formula generalizing (2.37) for the coefficients γ^i_{jk} to make the coordinate basis defined by (2.33) covariant constant at p is

$$\gamma^i_{jk} = \frac{1}{2}\sum_{p=1}^{N} g^{ip}\left\{g_{pj,k} + g_{pk,j} - g_{jk,p}\right\}, \tag{2.64}$$

where g^{ij} is the inverse matrix to g_{ij}. Then from (2.37) it is easy to deduce the local coordinate expression for the covariant derivative of a 1-form. Namely by writing

$$\alpha = \sum_{i=1}^{N} \alpha_i dz^i = \sum_{i=1}^{N} \alpha_i (dz^i + \sum_{j,k=1}^{N} \gamma^i_{jk} dz^j z^k) - \sum_{i,j,k=1}^{N} \alpha_i \gamma^i_{jk} dz^j z^k \tag{2.65}$$

and using (2.57) the familiar expression

$$\nabla\alpha = \sum_{i,j=1}^{N} \frac{\partial \alpha_i}{\partial z^j} dz^j \otimes dz^i - \sum_{i,j,k=1}^{N} \alpha_i \gamma^i_{jk} dz^j \otimes dz^k \tag{2.66}$$

results. The γ^i_{jk} are called the Christoffel symbols and are defined in any local coordinate system by the identity

$$\nabla_{\frac{\partial}{\partial z^i}} dz^j = -\sum_{k=1}^{N} \gamma^j_{ik} dz^k. \tag{2.67}$$

They are not the coefficients of a tensor.

EXERCISE 2.26. Show that the formula for covariant differentiation of a p-form is determined by

$$\nabla_{\frac{\partial}{\partial x^i}} dx^I = -\sum_{l=1}^{p}\sum_{k=1}^{N} \gamma^{I_l}_{ik} dx^{I_1} \wedge \ldots dx^{I_{l-1}} \wedge dx^k \wedge dx^{I_{l+1}} \wedge \cdots \wedge dx^{I_p}.$$

The identity (2.58) for covariant differentiation means that if V and W are $\mathcal{C}^\infty$ vector fields on X then for any connection on a vector bundle

$$[\nabla_V, \nabla_W](f\mu) - \nabla_{[V,W]}(f\mu) = f\left([\nabla_V, \nabla_W]\mu - \nabla_{[V,W]}\mu\right)$$
$$\forall\ f \in \mathcal{C}^\infty(X), \mu \in \mathcal{C}^\infty(X; E).$$

Thus

$$K_E(V,W)\mu = \left([\nabla_V, \nabla_W] - \nabla_{[V,W]}\right)\mu \tag{2.68}$$

is actually a differential operator of order 0, that is $K_E(V,W)$ is a section of the homomorphism bundle of E. It is called the curvature operator.

For the Levi-Civita connection acting on 1-forms, observe from (2.66) and (2.67) that the curvature operator evaluated on a coordinate basis is

$$\begin{aligned} &K_{A^1}(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k})dz^p \\ &= -\sum_q \left(\frac{\partial}{\partial z^j}\gamma^p_{kq} - \frac{\partial}{\partial z^k}\gamma^p_{jq}\right)dz^q + \sum_r\sum_q \left(\gamma^q_{jr}\gamma^p_{kq} - \gamma^q_{kr}\gamma^p_{jq}\right)dz^r. \end{aligned} \tag{2.69}$$

If the coordinates are chosen so that (2.35) holds then the coefficients γ^i_{jk} vanish at p and

$$K_{A^1}(p; \frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k})dz^p = \sum_q \left(\frac{\partial}{\partial z^k}\gamma^p_{jq} - \frac{\partial}{\partial z^j}\gamma^p_{kq}\right)dz^q.$$

LEMMA 2.27. *The curvature operator on T^*X of the Levi-Civita connection is*

$$K_{A^1}(p; \frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k})dz^l = \sum_{s,q=1}^N g^{ls}R_{jkqs}dz^q, \tag{2.70}$$

where R is the Riemann curvature tensor.

PROOF: Consider the local section of $\Phi\colon u \longrightarrow \widetilde{F}$ given by the orthonormalized coordinate differentials, ϕ^j in (2.38), i.e. $\Phi(x) = (\phi^1, \dots, \phi^N)$. Let $\widetilde{\gamma}_{ij} = \Phi^*\omega_{ij}$ be the pull-back of the connection forms under this section, so $\widetilde{\gamma}_{ij} = \sum\limits_{k=1}^N \widetilde{\gamma}^i_{jk}\phi^k$. Recalling that the ϕ^j are equal to the dz^j up to quadratic terms at p, the pull-back of the structure equation (2.48) then shows that

$$d\widetilde{\gamma}_{ij} = \sum_{pq} R_{ijpq}dz^p \wedge dz^q.$$

Since the pull-back of the form α_j is just ϕ^j it follows from (2.46) that (2.70) holds in this case since $g(p)$ is the Euclidean metric. As both sides of (2.70) are tensors it must hold in general.

EXERCISE 2.28. Find the curvature operator of the Levi-Civita connection on the tangent bundle in terms of the Riemann curvature tensor.

Combining (2.69) and (2.70) gives a formula for the Riemann curvature tensor, at any point p, computed in coordinates in which (2.35) holds, namely:

$$R_{ijkl} = \frac{1}{2}\left(-\frac{\partial}{\partial z^j}\frac{\partial}{\partial z^l}g_{ki} + \frac{\partial}{\partial z^j}\frac{\partial}{\partial z^k}g_{il} + \frac{\partial}{\partial z^i}\frac{\partial}{\partial z^l}g_{kj} - \frac{\partial}{\partial z^i}\frac{\partial}{\partial z^k}g_{jl}\right).$$

This shows that the curvature tensor has the following symmetries.

LEMMA 2.29. *The Riemann curvature tensor satisfies the identities:*

$$\begin{gathered} R_{ijkl} = -R_{jikl} \\ R_{ijkl} = R_{klij} \\ R_{ijkl} + R_{iklj} + R_{iljk} = 0 \quad \text{(1st Bianchi identity).} \end{gathered}$$

Since they are coordinate-invariant they hold in any coordinates.

2.10. Warped products.

For later application the connection and curvature for a simple warped product will be computed.

Thus let M be a Riemann manifold with metric h and let $\phi \in \mathcal{C}^\infty(\mathbb{S}^1)$ be a real-valued function on the circle. On $X = \mathbb{S}^1 \times M$ consider the metric

$$g = d\theta^2 + e^{2\phi(\theta)}h, \tag{2.71}$$

where θ is the standard variable on $\mathbb{S}^1$. Let ∇ be the covariant derivative, acting on 1-forms, on M. We shall also let ∇ denote the covariant derivative on the product when $\phi \equiv 0$ since then, with respect to the decomposition

$$T^*X = T^*\mathbb{S}^1 \oplus T^*M,$$

the covariant derivative with respect to $\partial/\partial\theta$ just acts on the coefficients and the covariant derivative with respect to a vector field on M acts through the connection of h.

LEMMA 2.30. *The Levi-Civita connection, $\widetilde{\nabla}$, of the metric g in* (2.71) *is given in terms of the product connection through*

$$\begin{gathered} \widetilde{\nabla}_{\partial_\theta} d\theta = 0,\ \widetilde{\nabla}_{\partial_\theta}\beta = -\frac{\partial\phi}{\partial\theta}\beta \\ \widetilde{\nabla}_V d\theta = \frac{\partial\phi}{\partial\theta}e^{2\phi}\mu, \\ \widetilde{\nabla}_V\beta = \nabla_V\beta - \frac{\partial\phi}{\partial\theta}\beta(V)d\theta \end{gathered} \tag{2.72}$$

for any $\mathcal{C}^\infty$ 1-form β on M and $\mathcal{C}^\infty$ vector field V on M, with μ the dual form to V, i.e. $\mu = h(V, \cdot)$.

Proof: From (2.64) the Christoffel symbols $\tilde{\gamma}$ for g can be computed for coordinates $z^0 = \theta$ and z^j, $j = 1, \dots, n = \dim M$ in M in terms of the Christoffel symbols, γ, of h :

$$\begin{aligned}\tilde{\gamma}^i_{jk} &= \gamma^i_{jk},\ \tilde{\gamma}^0_{jk} = -\frac{\partial\phi}{\partial\theta}e^{2\phi}h_{jk}\\ \tilde{\gamma}^i_{0k} = \tilde{\gamma}^i_{k0} &= \frac{\partial\phi}{\partial\theta}\delta^i_k,\ \tilde{\gamma}^0_{0k} = \tilde{\gamma}^0_{k0} = \tilde{\gamma}^0_{00} = 0.\end{aligned} \tag{2.73}$$

From these formulæ, and (2.67), (2.72) follows directly.

Lemma 2.31. *The curvature matrix $\tilde{R}$ of the metric* (2.71) *is given in terms of the curvature of M through*

$$\begin{aligned}\tilde{R}_{kj*0} = 0,\ \tilde{R}_{0j0k} &= h_{jk}\left(\frac{\partial^2\phi}{\partial\theta^2} - \left(\frac{\partial\phi}{\partial\theta}\right)^2\right)\\ \tilde{R}_{kjpq} &= e^{2\phi}R_{kjpq} + \left(\frac{\partial\phi}{\partial\theta}\right)^2 e^{4\phi}\left(h_{kp}h_{jq} - h_{kq}h_{jq}\right)\end{aligned} \tag{2.74}$$

where $j, k, p, q = 1, \dots, n = \dim M$.

Proof: This follows by assiduous use of (2.70) and Lemma 2.30.

2.11. Curvature formulæ.

If E' is the dual bundle to E then the transpose of a linear map defines an isomorphism

$$\hom(E) \equiv \hom(E'),\ A \longmapsto A^t.$$

Applying (2.61), repeatedly, shows that the curvature of the connection on the dual bundle satisfies

$$\begin{aligned}\langle K_{E'}(V,W)\mu',\mu\rangle + \langle\mu', K_E(V,W)\mu\rangle = 0\\ \forall\ \mu \in \mathcal{C}^\infty(X;E), \mu' \in \mathcal{C}^\infty(X;E').\end{aligned}$$

It follows easily that the curvature operator satisfies

$$K_{E'} = -K^t_E \tag{2.75}$$

(which provides a solution to Exercise 2.28). Similarly if E and F are two vector bundles with connections then (2.62) fixes a connection on $E \otimes F$. The homomorphism bundle of the tensor product is

$$\hom(E \otimes F) \equiv \hom(E) \otimes \hom(F).$$

In terms of this isomorphism a similar computation shows the curvature operator to be

$$K_{E\otimes F} = K_E \otimes \mathrm{Id} + \mathrm{Id} \otimes K_F. \tag{2.76}$$

As a special case the homomorphism bundle of E itself acquires a connection through the identity

$$\nabla(Au) = (\nabla A)u + A(\nabla u). \tag{2.77}$$

This is the same as the connection induced by the identification $\hom(E) = E \otimes E'$. As an operator on $\hom(E)$, the curvature is therefore given by

$$K_{\hom(E)}(V,W) = K_E(V,W) \cdot - \cdot K_E(V,W), \tag{2.78}$$

for any vector fields V, W, where the dots indicate that the two terms act by composition on the left and right respectively, i.e.

$$\begin{aligned} K_{\hom(E)}(V,W)A =& K_E(V,W)A - AK_E(V,W) \\ =& [K_E(V,W), A] \quad \forall\ A \in \hom(E). \end{aligned}$$

It is also important later to consider the 'big' homomorphism bundle, $\mathrm{Hom}(E)$, since this arises whenever operators on sections of E are considered. If E is a bundle over X this is the bundle over X^2 with fibre $\hom(E_{x'}, E_x)$ at (x, x'). This is sometimes called the exterior tensor product:

$$\mathrm{Hom}(E) = E \boxtimes E' \text{ over } X^2, \tag{2.79}$$

meaning that E should be pulled back to X^2 from the left factor of X and E' from the right, i.e.

$$\mathrm{Hom}(E) = \pi_L^* E \otimes \pi_E^* E' \text{ over } X^2, \tag{2.80}$$

where π_L and π_R are the two projections from X^2 to X onto the left and right factors.

Recall that the pull-back of a bundle E over Y under a smooth map $f \colon X \longrightarrow Y$ is always a bundle over X with fibre at x just $E_{f(x)}$. A section of E, $\mu \in \mathcal{C}^\infty(Y; E)$, defines a section of f^*E, namely $f^*\mu = \mu \circ f$. In fact every section of f^*E is locally a sum of products of $\mathcal{C}^\infty$ functions on X and these pulled-back sections:

$$\mathcal{C}^\infty(X; f^*E) = \mathcal{C}^\infty(X) \cdot f^*\mathcal{C}^\infty(Y; E). \tag{2.81}$$

This allows a connection on E to be pulled back to a connection on f^*E simply by defining

$$\begin{gathered} \nabla_{f^*E}\mu = \sum_j (dh_j \otimes f^*\mu_j + h_j f^*(\nabla_E \mu_j)) \\ \text{if } \mu = \sum_j h_j f^*\mu_j,\ h_j \in \mathcal{C}^\infty(X),\ \mu_j \in \mathcal{C}^\infty(Y; E). \end{gathered} \tag{2.82}$$

Thus sections covariant constant at $y \in Y$ pull back to be covariant constant at each $x \in f^{-1}(y)$.

LEMMA 2.32. *If $f\colon X \longrightarrow Y$ is a $\mathcal{C}^\infty$ map and E is a bundle over Y with connection and curvature operator K_E then the action of the curvature operator of the pulled-back connection on f^*E is*

$$(2.83)\qquad \begin{gathered} K_{f^*E}(V,W)\mu = K_E(f_*V, f_*W)\mu \in E_{f(x)} = (f^*E)_x \\ \forall\ V,W \in T_xX,\ \mu \in (f^*E)_x, \end{gathered}$$

i.e. if K_E is regarded as a section of $\Lambda^2X \otimes \hom(E)$ then $K_{f^*E} = f^*K_E$.

PROOF: It suffices to compute the curvature operator acting on covariant constant sections at a given point $x \in X$. In fact it is enough to consider sections $f^*\mu'$ with $\mu' \in \mathcal{C}^\infty(Y;E)$ covariant constant at $f(x)$. The second term in (2.68) is therefore zero at x and the first evaluates at x, using (2.82), to $f^*([\nabla_v,\nabla_w]\mu')$, where v, w are any $\mathcal{C}^\infty$ vector fields on Y with $v = f_*V$ and $w = f_*W$ at x. This gives (2.83).

The decomposition (2.80) of $\operatorname{Hom}(E)$ shows that it has a natural connection. Applying Lemma 2.32, (2.76) and (2.75) shows that the curvature operator for this connection is

$$(2.84)\quad K_{\operatorname{Hom}(E)}(V,W) = K_E\big((\pi_L)_*V,(\pi_L)_*W\big)\cdot - \cdot K_E\big((\pi_R)_*V,(\pi_R)_*W\big);$$

again the dots indicate that the terms act as operators on the left and right.

EXERCISE 2.33. Check that the connection on $\operatorname{Hom}(E)$ is consistent with that on $\hom(E)$ in the following sense. Let $\iota\colon \Delta \hookrightarrow X^2$ be the embedding of the diagonal, the set of points (x,x). Projection onto either factor of X gives the same diffeomorphism $\Delta \equiv X$. Thus pulling back $\operatorname{Hom}(E)$ under i gives a bundle over X. Show that this bundle is canonically isomorphic to $\hom(E)$ and that the pulled-back connection is the one defined above.

2.12. Orientation.

As noted already, the coframe bundle, a principal bundle with structure group $\mathrm{Gl}(N,\mathbb{R})$, of any compact manifold has a reduction to a principal $\mathrm{O}(N)$-bundle; namely the orthonormal coframe bundle, F, for any Riemann metric. In fact they were defined above in the other order. Each fibre, F_x, has two components. The manifold is orientable if and only if F itself has two components (assuming X to be connected). In this case either of the components gives a reduction of the structure group to $\mathrm{SO}(N)$. Thus an orientation, ${}_oF \subset F$, is simply a subbundle of the orthonormal frame bundle which is a principal $\mathrm{SO}(N)$-bundle, this subbundle is then the bundle of oriented orthonormal frames.

If a Riemann manifold is orientable and is given an orientation then the Riemann density becomes a form of maximal degree. Thus if ϕ^j, $j = 1,\dots,N$, is an oriented orthonormal frame in F_p then

$$\upsilon = \phi^1 \wedge \cdots \wedge \phi^N \in \Lambda^N_p(X)$$

is the volume form at p. It is independent of the choice of oriented orthonormal frame. The choice of orientation also fixes the Hodge (star) isomorphism:

$$\star\colon \Lambda^j_p(X) \longleftrightarrow \Lambda^{N-j}_p(X),$$

for each $p \in X$ and each j. If ϕ^j is an oriented orthonormal frame then

$$\star(\phi^{i_1} \wedge \cdots \wedge \phi^{i_j}) = \sigma \phi^{i_{j+1}} \wedge \cdots \wedge \phi^{i_N} \tag{2.85}$$

where σ is the sign of the permutation $(1,\dots,N) \longleftrightarrow (i_1,\dots,i_N)$. The forms on the left give a basis for $\Lambda^j_p(X)$, so $\star$ is defined by extending it linearly. The spaces $\Lambda^j(X)$ carry fibre metrics derived from the metric, on T^*X, by demanding that $\phi^1 \wedge \cdots \wedge \phi^j$ should have length 1 for any orthonormal basis of T^*_pX. In terms of this inner product on the form bundles, the operator (2.85) satisfies

$$\langle \star w, v\rangle \upsilon = w \wedge v. \tag{2.86}$$

In fact this shows that it is uniquely determined, independent of choice of oriented frame. It is therefore a $\mathcal{C}^\infty$ bundle map.

Moreover it follows from (2.85) that

$$\star^2 = (-1)^{j(N-j)} \text{ on } \Lambda^j(X). \tag{2.87}$$

The identity (2.86) makes $\star$ useful in analyzing the adjoint, δ, of d. The adjoint is defined by requiring

$$\int_X \langle \delta w, v\rangle \upsilon = \int_X \langle w, dv\rangle \upsilon,\ \forall\ w \in \mathcal{C}^\infty(X;\Lambda^j),\ \ v \in \mathcal{C}^\infty(X;\Lambda^{j-1}).$$

Using Stokes' theorem and (2.86) twice gives, for any $\mathcal{C}^\infty$ k form u and $N-k-1$ form v,

$$\int_X \langle \star dw, v\rangle \upsilon = \int_X dw \wedge v = (-1)^{k+1} \int_X w \wedge dv = (-1)^{k+1} \int_X \langle \star w, dv\rangle \upsilon.$$

Then using (2.87) it follows readily that

$$\delta = (-1)^{Nj+N+1} \star d \star \text{ on } \mathcal{C}^\infty(X;\Lambda^j). \tag{2.88}$$

Whilst the Hodge star operator is real it is useful to consider, in the even dimensional case, the complex operator

$$\tau = i^{j(j-1)+k}\star\colon \Lambda^j(X) \longrightarrow \Lambda^{2k-j}(X),\ \dim X = 2k. \tag{2.89}$$

By direct computation one then finds:

LEMMA 2.34. *If* $\dim X = 2k$ *is even then* τ *is an involution,* $\tau^2 = \mathrm{Id}$, *and*

$$\tau(d+\delta) + (d+\delta)\tau = 0.$$

In the odd-dimensional case there is a similar operator,

$$\tau' = i^{j(j-1)+k} \star R_\Lambda : \Lambda^j(X) \longrightarrow \Lambda^{2k-1-j}(X),\ \dim X = 2k-1,$$

where R_Λ is the parity involution

$$R_\Lambda = \begin{cases} 1 & \text{on } \Lambda^{\mathrm{evn}} \\ -1 & \text{on } \Lambda^{\mathrm{odd}}. \end{cases} \tag{2.90}$$

2.13. b-connections.

The connection on $\widetilde{F}$ can be recovered from (2.66) by noting that a frame is covariant constant at a point p if and only if each element of it is covariant constant. Then (2.67) can be used to extend the discussion to b-metrics. If g is a general b-metric on a manifold with boundary the problem is to determine what happens to the $C_{p,\phi}$ as p approaches the boundary. In the interior, but near the boundary, local coordinates

$$z^0 = \log x, z^j = y_j,\ j \geq 1, \tag{2.91}$$

can be used, where $x, y_1, \dots, y_n$ is a coordinate system up to the boundary and the dimension is now taken to be $N = n+1$. By definition the metric is smooth (in x, y) and non-degenerate with respect to the induced basis of ${}^bT^*X$,

$$g = \sum_{i,j=0}^{n} g_{ij}(x,y) dz^i dz^j,\ dz^0 = \frac{dx}{x},\ dz^j = dy_j,\ j \geq 1. \tag{2.92}$$

Using the formulae (2.64) for the Christoffel symbols, noting that $\partial_{z^0} = x\partial/\partial x$ while $\partial_{z^j} = \partial_{y_j}$ for $j \geq 1$ and (2.66) for the covariant derivative gives:

LEMMA 2.35. *The covariant derivative for a general b-metric on a manifold with boundary, X, extends from the interior to a first order differential operator:*

$$\begin{gathered} {}^b\nabla : \mathcal{C}^\infty(X; {}^b\Lambda^1) \longrightarrow \mathcal{C}^\infty(X; {}^bT^*X \otimes {}^b\Lambda^1), \\ {}^b\nabla \in \mathrm{Diff}^1_b(X; {}^b\Lambda^1, {}^bT^*X \otimes {}^b\Lambda^1) \\ {}^b\nabla(f\mu) = f{}^b\nabla\mu + df \otimes \mu\ \forall\ f \in \mathcal{C}^\infty(X), \mu \in \mathcal{C}^\infty(X; {}^bT^*X). \end{gathered} \tag{2.93}$$

In the last equation in (2.93), df should be interpreted as a b-form, really ${}^{b}df$.

PROOF: Applying (2.67) to the coordinates (2.91) gives

$$ {}^{b}\nabla dz^{j} = -\sum_{ik} \gamma^{j}_{ik} dz^{i} \otimes dz^{k}, $$

where the Christoffel symbols are given by (2.64). Since g_{ij} in (2.92) is non-degenerate, $g_{ij,0} = x\partial g_{ij}/\partial x$, and $g_{ij,l} = \partial g_{ij}/\partial y_l$ for $l > 0$, the γ^{j}_{ik} are indeed $\mathcal{C}^{\infty}$ down to $x = 0$. Thus the ${}^{b}\nabla dz^{j}$ are smooth sections of ${}^{b}T^{*}X \otimes {}^{b}\Lambda^{1}$ over the coordinate patch. The third part of (2.93) then follows by continuity and in turn leads to the other two parts.

Reversing the arguments above shows that an affine b-connection in the sense of (2.93) fixes the condition that a general b-frame, i.e. smoothly varying basis of ${}^{b}T^{*}X$, should be covariant constant at a point. Let

$$ {}^{b}F \subset {}^{b}\widetilde{F} $$

be the orthonormal coframe bundle for the b-metric as a subbundle (a principal $\mathrm{O}(n+1)$-bundle) of the general b-coframe bundle. Thus ${}^{b}\widetilde{F}_{p}$ consists of all the bases of ${}^{b}T^{*}X$ and ${}^{b}F_{p}$ all the orthonormal bases. Both are manifolds with boundary and ${}^{b}F$ is compact if X is compact. It follows that the $(n+1)$-planes $C_{p,e}$ extend smoothly from the interior to define

$$ \begin{aligned} {}^{b}\widetilde{C}_{p,\phi} &\subset {}^{b}T_{(p,\phi)}{}^{b}\widetilde{F} \quad \forall\ (p,\phi) \in {}^{b}\widetilde{F} \\ {}^{b}C_{p,\phi} &\subset {}^{b}T_{(p,\phi)}{}^{b}F \quad \forall\ (p,\phi) \in {}^{b}F. \end{aligned} $$

This is the more sophisticated formulation of the metric, i.e. Levi-Civita, b-connection, where (2.40) (and its generalization to $\mathrm{Gl}(n+1,\mathbb{R})$) and the analogue of (2.41):

$$ {}^{b}C_{p,\phi} \oplus T_{\phi}{}^{b}F_{p} = {}^{b}T_{(p,e)}{}^{b}F $$

continue to hold by continuity.

EXERCISE 2.36. Show that the structure equation as in (2.48) holds with $Q \in \mathcal{C}^{\infty}({}^{b}F; {}^{b}\Lambda^{2}F \otimes \mathfrak{so}(n+1))$ and that in consequence the Riemann curvature tensor becomes

$$ R \in \mathcal{C}^{\infty}(X; {}^{b}\Lambda^{2}X \otimes {}^{b}\Lambda^{2}X). $$

Of course the symmetry properties in Lemma 2.29 continue to hold by continuity.

Now what happens if the metric is assumed to be an exact b-metric? Taking local coordinates (2.91) in which (2.11) holds, the metric tensor satisfies

$$\begin{aligned} g_{00},\ g^{00} &= 1 + O(x^2) \\ g_{0i},\ g^{0i} &= O(x),\ i = 1, \dots, n. \end{aligned}$$

Then for the Christoffel symbols, with respect to the coordinates z^i introduced above, (2.64) implies that

$$\gamma^i_{j0} = O(x),\ i, j = 0, \dots, n.$$

This means that (2.66) can be rewritten as

(2.94)
$$\begin{aligned} \nabla\alpha = &\sum_{i=0}^{n} \partial_x \alpha_i dz^i \otimes dx + \sum_{i=0}^{n}\sum_{j=1}^{n} \partial_{z^j}\alpha_i dz^i \otimes dz^j \\ &- \sum_{i,j=0}^{n} \alpha_i (\frac{1}{x})\gamma^i_{j0} dz^j dx - \sum_{i,j=0}^{n}\sum_{k=1}^{n} \alpha_i \gamma^i_{jk} dz^j dy^k \end{aligned}$$

and so proves:

PROPOSITION 2.37. *For an exact b-metric the Levi-Civita connection extends by continuity to a connection on the b-frame bundle, and equivalently to a covariant derivative*

(2.95)
$$\nabla : \mathcal{C}^\infty(X; {}^b\Lambda^1) \longrightarrow \mathcal{C}^\infty(X; T^*X \otimes {}^b\Lambda^1),\ \nabla \in \mathrm{Diff}^1(X; {}^b\Lambda^1, T^*X \otimes {}^b\Lambda^1).$$

EXERCISE 2.38. Does the Levi-Civita connection of an exact b-metric define a b-connection on T^*X? Does it define a connection on T^*X?

Consider in a little more detail exactly what happens at the boundary in the case of an exact b-metric. The b-1-form dx/x is well-defined at ∂X as the section of length one of the orthocomplement in ${}^bT^*_{\partial X}X$ to $T^*\partial X$. If $F(\partial X)$ is the orthonormal frame bundle for the metric h induced on the boundary by the exact b-metric (2.11) then extending an orthonormal basis by adding dx/x to it gives the embedding

$$F(\partial X) \hookrightarrow {}^bF_{\partial X}(X).$$

This means that the structure group of ${}^bF(X)$, the orthonormal frame bundle, reduces to $\mathrm{O}(n)$ at ∂X. It is important to see the properties of the curvature of the b-metric which follow.

First note that the form dx/x, always the one appearing in (2.11) for the exact b-metric, satisfies

$$ {}^b\nabla\left(\frac{dx}{x}\right) = 0 \text{ in } {}^bT^*X \otimes {}^bT^*X \text{ at } \partial X. $$

Indeed this follows from (2.94). Furthermore, from (2.95),

$$ {}^b\nabla_{x\frac{\partial}{\partial x}}\alpha = 0 \text{ in } {}^bT^*X \otimes {}^bT^*X \text{ at } \partial X \quad \forall\ \alpha \in \mathcal{C}^\infty(X; {}^bT^*X). $$

Combining these observations with the commutation properties of vector fields tangent to the boundary easily gives:

PROPOSITION 2.39. *The Riemann curvature tensor of an exact b-metric is a* $\mathcal{C}^\infty$ *tensor in the usual sense:*

$$ R \in \mathcal{C}^\infty(X; \Lambda^2 \otimes \Lambda^2) $$

and has the property that its restriction to the boundary (in all arguments) is the Riemann curvature tensor of the induced metric on the boundary.

In this sense exact b-metrics correspond to truly cylindrical ends. The problem of the extension of all the discussion above, and the APS theorem itself, to the case of non-exact b-metrics is quite analogous to the recent discussion by Grubb [42] and Gilkey [38] of the extension of the more standard 'elliptic' formulation of the APS theorem to the case where the metric is not a product near the boundary.

PROOF: The commutator of any vector field tangent to the boundary with a vector field vanishing on the boundary also vanishes on the boundary, so the curvature operator on 1-forms vanishes at ∂X if either argument is $x\partial/\partial x$. This, and the symmetry properties, show that the curvature tensor is smooth in the ordinary sense, since any term containing dx/x vanishes at ∂X. It follows immediately from the definition as the curvature operator on 1-forms that the pull-back to the boundary of the Riemann tensor of the b-metric is the Riemann tensor of the boundary metric.

In general a b-connection on a vector bundle E is given by a first order differential operator ${}^b\nabla \in \mathrm{Diff}^1_b(X; E, {}^bT^*X \otimes E)$ which is a connection in the interior. It therefore satisfies (2.58) and so by continuity

$$ {}^b\nabla(f\mu) = df \otimes \mu + f({}^b\nabla\mu),\ f \in \mathcal{C}^\infty(X),\ \mu \in \mathcal{C}^\infty(X; E), \tag{2.96} $$

where df is, as in (2.93), interpreted as a section of ${}^bT^*X$. As for the Levi-Civita connection of an exact b-metric, one way to get such a b-connection is just to start with an ordinary connection and restrict its action to tangent vector fields. To characterize such connections amongst general b-connections, consider the dual sequence to (2.10):

$$0 \longrightarrow T^*\partial X \longrightarrow {}^bT^*_{\partial X}X \longrightarrow {}^bN^*\partial X \longrightarrow 0.$$

Here ${}^bN^*\partial X$ is a canonically trivial bundle over ∂X, with dx/x a section independent of the choice of defining function x. Projection onto this line therefore defines a differential operator, namely ${}^b\nabla_{x\partial/\partial x}$ evaluated at ∂X. The identity (2.96) shows that this is actually of order zero:

$$ {}^b\nabla_{x\partial/\partial x} \in \hom(E_{\partial X}). \tag{2.97}$$

PROPOSITION 2.40. *A b-connection on a bundle E over a compact manifold with boundary is (induced by) a connection if and only if the homomorphism ${}^b\nabla_{x\partial/\partial x}$ of $E_{\partial X}$ vanishes identically.*

PROOF: Certainly if ${}^b\nabla$ is induced by a connection ∇ then ${}^b\nabla_{x\partial/\partial x} = x\nabla_N$, where for some choice of defining function x the $\mathcal{C}^\infty$ vector field N satisfies $Nx = 1$ at ∂X. The homomorphism (2.97) then vanishes. Conversely if this homomorphism vanishes and N is such a normal vector field, any $\mathcal{C}^\infty$ vector field on X can be written in the form

$$W = V + fN,\ f \in \mathcal{C}^\infty(X),\ V \in \mathcal{V}_b(X).$$

The covariant derivative of a section e of E can then be defined by

$$\nabla_W e = {}^b\nabla_V + \frac{f}{x}({}^b\nabla_{xN}).$$

This defines a connection on E which induces the b-connection ${}^b\nabla$.

Note that if the homomorphism ${}^b\nabla_{x\partial/\partial x}$ is non-zero then the b-connection does not induce a connection on the boundary.

EXERCISE 2.41. Show that a section of ${}^bT^*X$ over X with the property that its restriction to ∂X projects to the zero section of ${}^bN^*\partial X$ is just a $\mathcal{C}^\infty$ section of T^*X. Use (2.96) to show that the difference between two b-connections is given by the tensor product with a $\mathcal{C}^\infty$ section of ${}^bT^*X$. Combine these two observations to give an alternative proof of Proposition 2.40.

Even when the homomorphism (2.97) for a b-connection is non-zero, it is still possible for the curvature operator to be a smooth form in the ordinary sense, rather than just a smooth b-form. Notice that ${}^b\nabla_V u$, for $V \in \mathcal{C}^\infty(\partial X; {}^bTX)$ and $u \in \mathcal{C}^\infty(\partial X; E)$, is a well-defined element of $\mathcal{C}^\infty(\partial X; E)$. From the discussion in §2.11 it follows that ${}^b\nabla_V F$ is similarly well defined if $F \in \mathcal{C}^\infty(\partial X; \hom(E))$.

LEMMA 2.42. *If X is a manifold with boundary and E is a vector bundle over X with b-connection then the curvature operator $K_E \in {}^b\Lambda^2 \otimes \hom(E)$ is a smooth form, i.e. $K_E \in \Lambda^2 \otimes \hom(E)$ if and only if the homomorphism (2.97) is covariant constant on the boundary in the sense that*

$$ {}^b\nabla_V({}^b\nabla_{x\partial/\partial x}) = 0 \ \forall\ V \in {}^bT_{\partial X}X. \tag{2.98}$$

PROOF: The b-connection on $\hom(E)$ satisfies (2.77), so if $A = {}^b\nabla_{x\partial/\partial x}$

$$({}^b\nabla_{x\partial/\partial x}A)u = {}^b\nabla_{x\partial/\partial x}(Au) - A{}^b\nabla_{x\partial/\partial x}u. \tag{2.99}$$

Thus ${}^b\nabla_{x\partial/\partial x}A = [A, A] = 0$. So (2.98) only depends on $V \in \mathcal{V}(\partial X)$. In fact (2.99) just corresponds to $K_E(x\partial/\partial x, x\partial/\partial x) = 0$, by antisymmetry. More generally if $V \in {}^bT_{\partial X}X$ then

$$\begin{aligned} K_E(x\partial/\partial x, V)u &= {}^b\nabla_{x\partial/\partial x}{}^b\nabla_V u - {}^b\nabla_V{}^b\nabla_{x\partial/\partial x}u - {}^b\nabla_{[x\partial/\partial x, V]}u \\ &= -({}^b\nabla_V A)u \end{aligned}$$

since $[x\partial/\partial x, V] = 0$ as a section of ${}^bT_{\partial X}X$. Thus (2.98) implies that, at ∂X, $K_E(x\partial/\partial x, \cdot) = 0$ and hence $K_E \in \Lambda^2 \otimes \hom(E)$ and conversely.

2.14. Characteristic classes.

No attempt is made to give a full treatment of characteristic classes here. However a brief discussion of the Weil homomorphism is included so that the integrands in (In.1) and (In.6) can be understood. See in particular [27] and [20]. Suppose that

$$h\colon \mathfrak{so}(N) \longrightarrow \mathbb{C} \tag{2.100}$$

is a function on $\mathfrak{so}(N)$ which is invariant under the adjoint action of $\mathrm{SO}(N)$, i.e.

$$h(GAG^{-1}) = h(A) \quad \forall\ A \in \mathfrak{so}(N),\ G \in \mathrm{SO}(N) \tag{2.101}$$

(or if X is not orientable under $\mathrm{O}(N)$.) The assumption that h is invariant, i.e. (2.101), means that the derivative satisfies

$$h'(A, [A, B]) = 0 \ \forall\ A, B \in \mathfrak{so}(N). \tag{2.102}$$

where $h'(A, B) = dh(A + tB)/dt$ at $t = 0$.

Consider the curvature form Q in (2.48) for X a compact manifold without boundary. It is a 2-form on F which takes values in $\mathfrak{so}(N)$. Alternatively it can be considered as an element of $\mathfrak{so}(N)$, the space of antisymmetric real 2×2 matrices, with entries which are 2-forms on F. The important point to note is that the exterior multiplication of 2-forms is commutative. Thus if h is a polynomial there is no ambiguity in evaluating $h(Q)$ as a form on F. In fact the wedge product of more than $N/2$ 2-forms vanishes, so even if h is a smooth function, $h(Q)$ is well defined as a 2-form on F by replacing h by at least the first $N/2$ terms in its Taylor series at 0. The point of the assumption, (2.101), of invariance of h is that (2.52) shows that

$$(A_O)^* h(Q) = h(OQO^t) = h(Q)$$

is actually invariant under the action of $\mathrm{SO}(N)$. A form on F which is invariant under the $\mathrm{SO}(N)$ action and vanishes if any one of its arguments is tangent to the fibre of $\pi\colon F \longrightarrow X$ is the lift of a uniquely defined form on the base.

EXERCISE 2.43. Show that if $\phi\colon U \longrightarrow F$ is a section over some open set of X and $h \in \mathcal{C}^\infty(F; \Lambda^* F)$ is invariant, $A_O^* h = h$ for all $O \in \mathrm{O}(N)$, and $h(v, \cdot, \cdots, \cdot) = 0$ if $v \in T_{\phi(p)} F_p$ then $\gamma = \phi^* h$ is a smooth form on U which is independent of the choice of ϕ and is such that $h = \pi^* \gamma$.

Thus there is a well-defined form $h(R)$ on X such that

$$h(Q) = \pi^*(h(R)),$$

$\pi\colon F \longrightarrow X$ being the bundle map. This form on the base represents the characteristic class associated to the invariant polynomial h and to the metric.

EXERCISE 2.44. Show that the notation here is a reasonable one, in that $h(R)$ expressed with respect to any orthonormal coframe ϕ^j is just h evaluated on $\sum\limits_{pq} R_{ijpq} \phi^p \wedge \phi^q$.

The structure equation (2.48) has the important consequence:

LEMMA 2.45. *The form $h(R)$ defined by an invariant polynomial* (2.100) *is always closed.*

PROOF: Since wedge product of 2-forms commutes with all forms, in particular 3-forms, the exterior derivative of $h(Q)$ can be written unambiguously as

$$dh(Q) = h'(Q, dQ), \tag{2.103}$$

where the derivative of h, $h'(Q, A)$, is a linear functional in $A \in \mathfrak{so}(N)$. From (2.48)

$$dQ_{ij} = \sum_q \left[d\omega_{iq} \wedge \omega_{qj} - \omega_{iq} \wedge d\omega_{qj}\right] = \sum_q \left[Q_{iq} \wedge \omega_{qj} - \omega_{iq} \wedge Q_{qj}\right].$$

since the terms with three factors of ω cancel. This is just the commutator in the Lie algebra $\mathfrak{so}(N)$ so

$$dQ = [Q, \omega] \quad \text{(2nd Bianchi identity)}. \tag{2.104}$$

Inserting (2.104) in (2.103) and using (2.102) shows that $h(R)$ is indeed a closed form.

The classic case of this construction gives the Pontriagin forms. These arise by taking the invariant polynomial

$$h(A) = \det(\mathrm{Id} + \frac{A}{2\pi});$$

the invariance being a consequence of the multiplicativity of the determinant. The kth Pontrjagin form is the term of degree $4k$ in $h(R)$; it is easy to see that the terms of degree $4k+2$ all vanish.

The de Rham theorem allows $h(R)$ to be interpreted as a cohomology class on X. It is important to note that this class is independent of the choice of metric. Indeed any two metrics, g^0 and g^1, can be connected by a smooth one-parameter family of metrics g^t. The construction of the connection is clearly smooth in the parameter, so the forms ω^t and Q^t depend smoothly on t. Differentiating the structure equation gives:

$$\frac{dQ^t_{ij}}{dt} = d\frac{d\omega^t_{ij}}{dt} + \sum_q \left[\frac{d\omega^t_{iq}}{dt} \wedge \omega^t_{qj} + \omega^t_{iq} \wedge \frac{d\omega^t_{qj}}{dt}\right].$$

Consider the one-parameter family of closed forms $h(Q^t)$ on F :

$$\begin{aligned}\frac{dh(Q^t)}{dt} =& h'(Q^t; \frac{dQ^t}{dt})\\ =& h'(Q^t; d\frac{d\omega^t}{dt}) + h'(Q^t; \frac{d\omega^t}{dt} \wedge \omega^t + \omega^t \wedge \frac{d\omega^t}{dt}).\end{aligned}$$

The first term here can be written $d\,[h'(Q^t; d\omega^t/dt)]$ since Q^t is closed. The second term is $\mathrm{O}(N)$-invariant and only ω^t is non-zero on vertical vectors, so evaluated on $v \in T_\phi F_p$, with $A = \omega^t(v)$,

$$h'(Q^t; \frac{d\omega^t}{dt} \wedge \omega^t + \omega^t \wedge \frac{d\omega^t}{dt})(v, \cdot) = h'(Q^t; [\frac{d\omega^t}{dt}, A])(\cdot) = 0$$

by (2.102). Since it also vanishes with ω^t, i.e. on the connection planes, this second term vanishes identically and hence

$$h(Q^1) - h(Q^0) = dT,\ T = \int_0^1 h'(Q^t, \frac{d\omega^t}{dt})dt. \tag{2.105}$$

This transgression formula shows that the de Rham class of $h(R)$, i.e. its equivalence class up to exact forms, is independent of the choice of metric used to define it. It therefore corresponds to a cohomology class.

The nilpotence of the exterior action of a 2-form means that to define $h(R)$ it suffices for the polynomial h to be obtained as the Taylor series of a function which is just defined near the origin in $\mathfrak{so}(N)$, provided it is invariant there. Such examples arise directly in the computation of the index form. In particular the $\widehat{A}$-genus:

$$\widehat{A}(X) = \det{}^{\frac{1}{2}}\left(\frac{R/4\pi i}{\sinh(R/4\pi i)}\right) \tag{2.106}$$

is a well-defined form, depending only on the metric and orientation. Here $\det^{\frac{1}{2}}$ is the positive square root of the determinant of a matrix. It is defined near the identity and is invariant under conjugation by $\mathrm{O}(N)$. Its argument is a formal power series with first term the identity matrix, so the square root of the determinant is a well-defined invariant formal power series.

EXERCISE 2.46. Show that $\widehat{A}(X)$ can be expressed in terms of the Pontrjagin forms.

Similarly the L-genus of Hirzebruch is

$$L(X) = \det{}^{\frac{1}{2}}\left(\frac{R/4\pi i}{\tanh(R/4\pi i)}\right). \tag{2.107}$$

It arises in the signature formula (see §9.3).

2.15. Hermitian bundles.

So far we have been dealing mainly with real bundles. In practice many of the examples below are complex. A connection on a complex bundle is automatically assumed to be complex, i.e. $\mathbb{C}$-linear. The existence of such a connection follows as in the discussion leading to Lemma 2.21.

A complex bundle is Hermitian if it carries a sesqui-linear inner product. A connection on the bundle is unitary (or Hermitian) if it satisfies (2.61) with respect to this pairing, i.e.

$$V\langle e, f\rangle = \langle \nabla_V e, f\rangle + \langle e, \nabla_V f\rangle \tag{2.108}$$

for all real vector fields V.

EXERCISE 2.47. Show that the proof of Lemma 2.21 also implies that any Hermitian bundle has a unitary connection.

For Hermitian bundles with unitary connections there is a similar construction of characteristic classes as in the real case, leading to the Chern classes. The curvature operator

$$K_E(V,W) = [\nabla_V, \nabla_W] - \nabla_{[V,W]} \in \mathcal{C}^\infty(X; \hom(E))$$

takes values in the subbundle, $\mathfrak{u}(E) \subset \hom(E)$, formed by the Lie algebras of the unitary groups of the Hermitian structures. An invariant polynomial

$$h \colon \mathfrak{u}(m) \longrightarrow \mathbb{C},\ \ h(GAG^*) = h(A)\ \forall\ A \in \mathfrak{u}(m), G \in \mathrm{U}(m),$$

where m is the fibre dimension of E, fixes a bundle map $\mathcal{C}^\infty(X; \mathfrak{u}(E)) \longrightarrow \mathcal{C}^\infty(X)$. Regarding K_E as a 2-form with values in $\mathfrak{u}(E)$ it follows, as in the discussion above, that $h(K_E)$ is a well-defined closed form on X. Necessarily it has only terms of even degree.

For example the Chern forms, $\mathrm{cc}_j(E)$, of the bundle are the terms of degree $2j$ in the total Chern form

$$\mathrm{cc}(E) = \det\left(\mathrm{Id} + i\frac{K_E}{2\pi}\right).$$

There is a similar transgression formula showing that $\mathrm{cc}_j(E)$ is well defined as a cohomology class independent of the choice of Hermitian structure and unitary connection on E. The Chern character of the bundle is defined as the characteristic class

$$\mathrm{Ch}(E) = \mathrm{tr}\left[\exp\left(\frac{iK_E}{2\pi}\right)\right]. \tag{2.109}$$

The term which is homogeneous of degree k in $A \in \mathfrak{u}(2k)$ in the trace of the exponential

$$\mathrm{Pf}(A) = \frac{1}{k!}\frac{d^k}{dr^k}\,\mathrm{tr}\exp(rA) \tag{2.110}$$

is an invariant polynomial called the Pfaffian. It occurs in the Gauss-Bonnett formula (see §9.2).

The combination of (2.106) and (2.109) means that the Atiyah-Singer integrand in (In.2) is now defined. The integral in (In.1) picks out the term of maximal form degree, $\dim X$, and integrates it over the oriented manifold X.

2.16. de Rham cohomology.

Before briefly discussing the extension of the construction of characteristic classes to the 'b' setting, first consider the various forms of the de Rham theorem on a compact manifold with boundary. The standard complex, (2.17), yields the 'absolute' de Rham spaces:

$$\mathcal{H}^k_{\mathrm{dR,abs}}(X) = \left\{u \in \mathcal{C}^\infty(X;\Lambda^k); du = 0\right\} / d\mathcal{C}^\infty(X;\Lambda^{k-1}). \tag{2.111}$$

On the other hand the operator d also defines maps

$$d\colon \dot{\mathcal{C}}^\infty(X;\Lambda^k) \longrightarrow \dot{\mathcal{C}}^\infty(X;\Lambda^{k+1}),$$

where for any vector bundle E over a manifold with boundary (or corners) $\dot{\mathcal{C}}^\infty(X;E) \subset \mathcal{C}^\infty(X;E)$ is the subspace of section vanishing to all orders at the boundary. Thus $u \in \dot{\mathcal{C}}^\infty(X;E)$ if it is the restriction to X of some section $u' \in \mathcal{C}^\infty(\widetilde{X};\widetilde{E})$ which vanishes on $\widetilde{X} \setminus X$, for some extensions of X and E. The relative de Rham spaces are

$$\mathcal{H}^k_{\mathrm{dR,rel}}(X) = \left\{u \in \dot{\mathcal{C}}^\infty(X;\Lambda^k); du = 0\right\} / d\dot{\mathcal{C}}^\infty(X;\Lambda^{k-1}). \tag{2.112}$$

THEOREM 2.48. *The absolute and relative de Rham cohomology spaces of a compact manifold with boundary are canonically isomorphic to the (singular) cohomology spaces of the interior, with arbitrary and compact supports respectively.*

There is also a natural b-de Rham space. Namely one can take the cohomology of the complex (2.21):

$$\mathcal{H}^k_{\mathrm{dR,b}}(X) = \left\{u \in \mathcal{C}^\infty(X;{}^b\Lambda^k); {}^b du = 0\right\} / {}^b d\mathcal{C}^\infty(X;{}^b\Lambda^{k-1}), \tag{2.113}$$

where the notation ${}^b d$ is only briefly resuscitated for emphasis.

PROPOSITION 2.49. ([4]) *The b-de Rham spaces of a compact manifold decompose canonically:*

$$\mathcal{H}^k_{\mathrm{dR,b}}(X) = \mathcal{H}^{k-1}_{\mathrm{dR}}(\partial X) \oplus \mathcal{H}^k_{\mathrm{dR,abs}}(X). \tag{2.114}$$

PROOF (BRIEF): There is an obvious map at the level of forms from $\mathcal{H}^k_{\mathrm{dR,abs}}(X)$ into $\mathcal{H}^k_{\mathrm{dR,b}}(X)$ since closed smooth sections of $\Lambda^k X$ are naturally closed smooth sections of ${}^b\Lambda^k X$. At the level of forms the map from $\mathcal{H}^{k-1}_{\mathrm{dR}}(\partial X)$ is given in terms of a collar decomposition of X near ∂X, a

[4] Unpublished joint result with R.R. Mazzeo.

boundary coordinate x and $\phi \in \mathcal{C}^\infty(\mathbb{R})$ with $\phi(0) = 1$ and $\phi(x)$ supported in the collar by

$$\mathcal{C}^\infty(\partial X; \Lambda^{k-1}) \ni \alpha \longmapsto \phi(x)\frac{dx}{x} \wedge \alpha \in \mathcal{C}^\infty(X; {}^b\Lambda^k). \tag{2.115}$$

At the level of cohomology these maps give (2.114). The inverse to (2.115) is given by contraction with $x\partial/\partial x \in {}^bT_{\partial X}X$:

$$\mathcal{H}^k_{\mathrm{dR,b}}(X) \ni \beta \longmapsto \beta(x\frac{\partial}{\partial x}, \cdot) \in \mathcal{H}^{k-1}_{\mathrm{dR}}(\partial X).$$

2.17. *b*-characteristic classes.

Consider the alterations which must be made to the treatment of characteristic classes in §2.14 and §2.15 in the case of a b-metric or a b-connection.

The formulæ in §2.14 are local in nature and so continue to hold when interpreted as involving b-forms. In particular, with R the curvature tensor of a b-metric and h an invariant polynomial, $h(R)$ is a well-defined smooth closed b-form on X, which is now a compact manifold with boundary. Applying Proposition 2.49, such a form defines both an absolute cohomology class on X and a class on ∂X. If the metric is an exact b-metric then R is a smooth form and the second term vanishes, so $h(R)$ defines an absolute cohomology class.

On the other hand observe what happens in the transgression formula (2.105). In the case of a general b-metric the form T is a smooth b-form on F but, because of the presence of $d\omega^t/dt$, is not necessarily a smooth form. If the deformation is through exact b-metrics then the connection form is always smooth, so T will in fact be smooth. In any case it follows that the characteristic classes give well-defined absolute cohomology classes on X.

Thus the first term in (In.6) is defined as the integral over X, a compact manifold with boundary, of the volume form component of AS, given by (In.2). As distinct from the boundaryless case, this does not have direct topological significance, since it does not correspond to a pairing in cohomology (as absolute cohomology classes pair with relative homology classes). In particular under a deformation of the metric the two terms in (In.6) need not be separately preserved. The variation of the first (and hence of the second) can be obtained from the transgression formula, (2.105), and Stokes' theorem. This is discussed in more detail in §8.15.

Chapter 3. Spin structures

The twisted Dirac operators involved in the APS theorem, as stated in the Introduction, are defined in terms of a spin structure on the manifold. This is now discussed, as is the more general notion of a Dirac operator associated to a Clifford module with connection. Needless to say these concepts are extended to the case of compact manifolds with boundary equipped with b-metrics.

3.1. Euclidean Dirac operator.

Dirac, wanting to quantize the electron, looked for a first-order differential operator with square the d'Alembertian, i.e. the wave operator. This is just the Laplacian for a metric of signature $+,-,-,-$ and one can as well do the same thing for the ordinary Laplacian. The question is: Does there exist a matrix-valued constant coefficient operator

$$\eth^+ = \sum_{j=1}^{n} \gamma_j D_j, \quad D_j = \frac{1}{i}\frac{\partial}{\partial x_j} \tag{3.1}$$

such that

$$(\eth^+)^* \cdot \eth^+ = \sum_{j=1}^{n} D_j^2 = \Delta? \tag{3.2}$$

Here the adjoint is the usual matrix adjoint and the operator adjoint with respect to Lebesgue measure, i.e. $D_j^* = D_j$. Clearly (3.2) holds if and only if the matrices satisfy

$$\gamma_j^* \gamma_j = \mathrm{Id},\ \gamma_j^* \gamma_k + \gamma_k^* \gamma_j = 0,\ j \neq k,\ j,k = 1,\dots,n. \tag{3.3}$$

In one and two dimensions it is easy to solve (3.3). If $n = 1$, taking $\gamma_1 = i$ gives $\eth^+ = d/dx$. If $n = 2$, then $\gamma_1 = i$, $\gamma_2 = -1$ are 1×1 matrix solutions. So for $n = 2$ the "obvious" solution (leading to complex analysis and hence a good part of nineteenth century mathematics) is

$$\eth^+ = \bar\partial = (\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$$
$$\Longrightarrow (\eth^+)^* = -\partial = -(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}),\ \Delta = -\partial\bar\partial = \eth^{+*}\eth^+.$$

Following Dirac, (3.3) can be solved with 2×2 matrices for $n = 4$. Taking $\gamma_1 = i\,\mathrm{Id}$, as seems reasonable, the remaining conditions become

$$\gamma_j^* = \gamma_j,\ \gamma_j^2 = \mathrm{Id},\ \gamma_j\gamma_k + \gamma_k\gamma_j = 0, \quad j \neq k,\ j,k \geq 2. \tag{3.4}$$

Thus the remaining matrices are self-adjoint anticommuting involutions. It is then easy to guess a solution, for example

$$\gamma_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ \gamma_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ \gamma_4 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} = i\gamma_2\gamma_3. \tag{3.5}$$

EXERCISE 3.1. Check that these *Pauli matrices* do solve the problem (3.1), (3.2) for $n = 4$.

As will be shown below there is a solution of (3.1), (3.2) for $n = 2k$ amongst $2^{k-1} \times 2^{k-1}$ matrices. Of course having found a solution for any one n, it works for smaller values, simply by dropping some of the matrices. Note that the original idea was to find $\eth$ so that $\eth^2 = \Delta \cdot \mathrm{Id}$. If $\gamma_1 = i\,\mathrm{Id}$ and the γ_j for $j > 1$ satisfy (3.4) then $\gamma_1 = -i\,\mathrm{Id}$ and γ_j also satisfy (3.3). So setting $\eth^- = \eth_+^*$, the operator

$$\eth = \begin{bmatrix} 0 & \eth^- \\ \eth^+ & 0 \end{bmatrix} \text{ satisfies } \eth^2 = \Delta \cdot \mathrm{Id},\ \eth^* = \eth.$$

It will therefore follow that for $n = 2k, 2k+1$ there is a self-adjoint solution of (3.1) and (3.2) amongst operators with values in $2^k \times 2^k$ matrices (and no smaller).

The construction of solutions to (3.1), (3.2) in general will be through the Clifford algebra. To see how this arises, suppose $\eth$ is a $p \times p$ matrix-valued operator satisfying $\eth^2 = \Delta \cdot \mathrm{Id}$. The symbol of $\eth$ is a matrix depending linearly on $\xi \in \mathbb{R}^n$:

$$\sigma(\eth) = \sum_{i=1}^{n} \gamma_i \xi_i.$$

If $M_{\mathbb{C}}(p)$ is the algebra of complex $p \times p$ matrices then $\sigma(\eth)$ is a linear map

$$\mathbb{R}^n \ni \xi \longmapsto \sigma(\eth)(\xi) \in M_{\mathbb{C}}(p), \tag{3.6}$$

such that $\sigma(\eth)(\xi) \cdot \sigma(\eth)(\xi) = |\xi|^2\,\mathrm{Id}$.

This must be true for all ξ, so replacing ξ by $\xi + \eta$ and expanding gives

$$\sigma(\eth)(\xi) \cdot \sigma(\eth)(\eta) + \sigma(\eth)(\eta) \cdot \sigma(\eth)(\xi) = 2\langle \xi, \eta \rangle\,\mathrm{Id}. \tag{3.7}$$

This identity, derived for a linear map (3.6), makes perfectly good sense for a linear map

$$c : V \ni \xi \longmapsto \mathrm{cl}(\xi) \in A \tag{3.8}$$

if V is a (finite dimensional) vector space with an inner product $\langle , \rangle$ and A is an associative algebra with identity.

Any linear map, (3.8), extends to a map from the j-fold tensor product

$$\mathrm{Ten}_j(V) = V \overset{j \text{ factors}}{\otimes \cdots \otimes} V \longrightarrow A,$$
$$\sum a_{i_1 \dots i_j} e_{i_1} \otimes \cdots \otimes e_{i_j} \longmapsto \sum a_{i_1 \dots i_j} \operatorname{cl}(e_{i_1}) \dots \operatorname{cl}(e_{i_1}).$$

Thus such a map extends to the full tensor algebra

$$\hat{c}\colon \mathrm{Ten}(V) = \bigoplus_{j=0}^{\infty} \mathrm{Ten}_j(V) \longrightarrow A. \tag{3.9}$$

The identity (3.7) means that under (3.9) the ideal

$$\mathcal{I}_{\mathrm{Cl}}(V) = \Big\{ \sum_{\text{finite}} u_\alpha \big(\xi_\alpha \otimes \eta_\alpha + \eta_\alpha \otimes \xi_\alpha - 2\langle \xi_\alpha, \eta_\alpha \rangle \operatorname{Id} \big) v_\alpha; \quad \xi_\alpha, \eta_\alpha \in V, u_\alpha, v_\alpha \in \mathrm{Ten}(V) \Big\} \tag{3.10}$$

should be mapped to zero. Thus the quotient algebra

$$\mathrm{Cl}(V) = \mathrm{Ten}(V) \big/ \mathcal{I}_{\mathrm{Cl}}(V) \tag{3.11}$$

has the *universal property* that any linear map (3.8) satisfying[5]

$$\operatorname{cl}(\xi) \cdot \operatorname{cl}(\eta) + \operatorname{cl}(\eta) \cdot \operatorname{cl}(\xi) = 2\langle \xi, \eta \rangle \operatorname{Id} \tag{3.12}$$

extends to an algebra homomorphism

$$\tilde{c}\colon \mathrm{Cl}(V) \longrightarrow A.$$

3.2. Clifford algebra.

This makes it natural to study the *Clifford algebra* of V (with inner product $\langle , \rangle$) defined by (3.10), (3.11). The Clifford algebra is *filtered* by the order gradation from the tensor algebra:

$$\mathrm{Cl}^{(k)}(V) = \left\{ u \in \mathrm{Cl}(V); \exists\, v \in \bigoplus_{j=0}^{k} \mathrm{Ten}_j(V),\ u = [v] \right\}. \tag{3.13}$$

Thus the *order* of an element of $\mathrm{Cl}(V)$ is the minimum order of a representative in the tensor algebra. Note that the relation, (3.12), equates the

[5] Many authors use a different sign convention, so that the square of an element of V of unit length is $-\operatorname{Id}$. This corresponds to replacing the matrices γ_j by $i\gamma_j$.

symmetric part of an element of order 2 with an element of order 0. Thus the quotients are naturally identified:

$$\mathrm{Cl}^{(k)}(V)/\,\mathrm{Cl}^{(k-1)}(V) \equiv \Lambda^k(V),\ A_k\colon\ \mathrm{Cl}^{(k)}(V) \longrightarrow \Lambda^k(V),\ \forall\ k. \tag{3.14}$$

The map here is just the usual total antisymmetrization map, which means for example that if $c_1, c_2 \in V$ then

$$A_2(c_1c_2) = A_2(\frac{1}{2}(c_1 \cdot c_2 + c_2 \cdot c_1) + \frac{1}{2}[c_1, c_2]) = c_1 \wedge c_2.$$

The null space is the ideal generated by $\xi \otimes \eta + \eta \otimes \xi$, $\xi, \eta \in V$.

Let $e_1, \dots, e_n$ be an orthonormal basis of V. As a consequence of (3.14) the Clifford algebra $\mathrm{Cl}(V)$ has a basis given by the products in strictly increasing order:

$$e_{i_1} \cdot e_{i_2} \cdots e_{i_j},\ i_1 < i_2 < \cdots < i_j,\ j = 0, \dots, n = \dim V.$$

In particular

$$\dim_{\mathbb{R}} \mathrm{Cl}(V) = 2^{\dim V}. \tag{3.15}$$

Thus every element of $\mathrm{Cl}(V)$ has order at most $n = \dim V$:

$$\mathbb{R} \cdot \mathrm{Id} = \mathrm{Cl}^{(0)}(V) \subset \mathrm{Cl}^{(1)}(V) \subset \cdots \subset \mathrm{Cl}^{(n)}(V) = \mathrm{Cl}(V).$$

From (3.15) the dimension of $\mathrm{Cl}(V)$ is equal to that of the exterior algebra

$$\Lambda^* V = \bigoplus_{k=0}^{\dim V} \Lambda^k V.$$

In fact the exterior algebra arises as the Clifford algebra when the inner product is taken to vanish identically. From the form of the relation (3.12)

$$\mathrm{Cl}^{(1)}(V) \equiv V \oplus (\mathbb{R} \cdot \mathrm{Id}) = V \oplus \mathrm{Cl}^{(0)}(V). \tag{3.16}$$

There is another close relationship between $\mathrm{Cl}(V)$ and $\Lambda^* V$, namely an action of $\mathrm{Cl}(V)$ on $\Lambda^* V$:

$$c\colon\ \mathrm{Cl}(V) \longrightarrow \hom(\Lambda^* V). \tag{3.17}$$

In (3.16), the identity is mapped under (3.17) to the identity whereas

$$\mathrm{cl}(\xi) = i\,[\mathrm{ext}(\xi) - \mathrm{int}(\xi)] : \Lambda^* V \longrightarrow \Lambda^* V. \tag{3.18}$$

The *contraction* or inner product with ξ, $\operatorname{int}(\xi)$, is defined by evaluation on the vector dual to ξ,

$$v_\xi \in V^*, \quad v_\xi(w) = \langle \xi, w \rangle \quad \forall\ w \in V.$$

Then $\alpha \in \Lambda^k V$ can be identified as a totally antisymmetric multilinear map and if

$$\begin{gathered} \alpha \colon V^* \otimes \cdots \otimes V^* \longrightarrow \mathbb{R}, \\ \operatorname{int}(\xi)\alpha = k!\alpha(v_\xi, \dots, .) \in \Lambda^{k-1}V. \end{gathered} \tag{3.19}$$

The exterior product $\operatorname{ext}(\xi)$ is just wedge product (on the left) with ξ :

$$\operatorname{ext}(\xi)\alpha = \xi \wedge \alpha.$$

EXERCISE 3.2. Show, using (2.70), that the action of the curvature operator of the Levi-Civita connection on the exterior algebra can be written in terms of the Riemann curvature tensor as

$$\sum_{ij} dz^i \wedge dz^j \sum_{kl} R_{ijlk} \operatorname{ext}(dz^k) \operatorname{int}(dz^l). \tag{3.20}$$

Clifford multiplication, (3.18), extends to an algebra homomorphism (3.17) since, essentially by definition, it extends to the tensor algebra, as a map into the algebra $\hom(\Lambda^* V)$ and from (3.18), (3.19)

$$\operatorname{cl}(\xi) \cdot \operatorname{cl}(\xi) = \operatorname{ext}(\xi) \operatorname{int}(\xi) + \operatorname{int}(\xi) \operatorname{ext}(\xi) = |\xi|^2.$$

Thus c projects to (3.17).

So far the coefficients in the Clifford algebra have been real, i.e. the tensor product in (3.9) is over $\mathbb{R}$. Even when V itself is real, it is useful to consider the *complexified* Clifford algebra

$$\mathbb{C}\mathrm{l}(V) = \mathbb{C} \otimes_{\mathbb{R}} \mathrm{Cl}(V). \tag{3.21}$$

This just arises by taking complex coefficients in the definition of the tensor algebra and the ideal. Since the coefficient matrices, γ_i, are expected to be complex it is reasonable to consider (3.21). In any case the structure of the complexified algebra is simpler.

3.3. Periodicity.

For each integer $k \geq 1$ let $\mathrm{Cl}(k)$ be the Clifford algebra generated by $\mathbb{R}^k$ with the Euclidean inner product, so $\mathbb{C}\mathrm{l}(k)$ is its complexification. Similarly, $M_{\mathbb{C}}(r)$ denotes the complexification of $M(r)$, i.e. the algebra of complex $r \times r$ matrices. The fundamental *periodicity* result is:

Lemma 3.3. *For any $k \in \mathbb{N}$*

$$\mathbb{C}l(k+2) \simeq \mathbb{C}l(k) \otimes M_{\mathbb{C}}(2). \tag{3.22}$$

Proof: Let $\xi_1, \dots, \xi_{k+2}$ be the natural basis of $\mathbb{R}^{k+2}$. It certainly suffices to show that there is a linear map

$$\mathbb{R}^{k+2} \longrightarrow \mathbb{C}l(k) \otimes M_{\mathbb{C}}(2)$$

which extends to give the isomorphism (3.22). Since $\xi_1, \dots, \xi_k$ can be taken as the canonical basis of $\mathbb{R}^k$, it makes sense to define

$$\xi_j \longmapsto \xi_j \otimes \gamma_2, \quad j = 1, \dots, k \tag{3.23}$$

and for the last two generators to take the other Pauli matrices:

$$\begin{aligned} \xi_{k+1} &\longmapsto \mathrm{Id} \otimes \gamma_3 \\ \xi_{k+2} &\longmapsto \mathrm{Id} \otimes \gamma_4, \end{aligned}$$

with the γ_i fixed by (3.5). Of course $\mathrm{Id} \in \mathbb{C}l(k+2)$ is to be sent to $\mathrm{Id} \in \mathbb{C}l(k) \otimes M_{\mathbb{C}}(2)$. Since it is defined on the generators this gives an algebra homomorphism $\mathrm{Ten}_{\mathbb{C}}(\mathbb{R}^{k+2}) \longrightarrow \mathbb{C}l(k) \otimes M_{\mathbb{C}}(2)$. To check that it descends to $\mathbb{C}l(k+2)$ consider the generating relations. Certainly from (3.23)

$$\xi_j \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \xi_l \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \xi_j \cdot \xi_l \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad j, l = 1, \dots, k.$$

The remaining relations follow from the properties of the Pauli matrices in (3.4). This gives an algebra homomorphism

$$\mathbb{C}l(k+2) \longrightarrow \mathbb{C}l(k) \otimes M_{\mathbb{C}}(2). \tag{3.24}$$

To see that this map is surjective, notice that off-diagonal elements can be obtained as follows

$$\begin{aligned} \alpha_e \cdot \xi_{k+1} &\longmapsto \alpha_e \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ \alpha_o \cdot \xi_{k+1} &\longmapsto \alpha_o \otimes \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ \alpha_e \cdot \xi_{k+2} &\longmapsto \alpha_e \otimes \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \\ \alpha_o \cdot \xi_{k+2} &\longmapsto \alpha_o \otimes \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \end{aligned} \tag{3.25}$$

where α_e is a sum of multiples of products of even numbers of the elements $\xi_1, \ldots, \xi_k$ and similarly α_o is a sum of multiples of products of odd numbers of these basis elements. Similarly the diagonal elements can be obtained by noting that

$$\begin{aligned} \alpha_e &\longmapsto \alpha_e \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \alpha_o &\longmapsto \alpha_o \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ \alpha_e \cdot \xi_{k+1} \cdot \xi_{k+2} &\longmapsto \alpha_e \otimes \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \\ \alpha_o \cdot \xi_{k+1} \cdot \xi_{k+2} &\longmapsto \alpha_o \otimes \begin{bmatrix} -i & 0 \\ 0 & -i \end{bmatrix}. \end{aligned} \tag{3.26}$$

This also shows the injectivity of the map (3.24), proving the lemma.

As a direct consequence of this isomorphism

$$\mathbb{C}\mathrm{l}(2k) \simeq M_{\mathbb{C}}(2^k) \tag{3.27}$$

$$\mathbb{C}\mathrm{l}(2k+1) \simeq M_{\mathbb{C}}(2^k) \oplus M_{\mathbb{C}}(2^k). \tag{3.28}$$

Indeed the block decomposition of matrices shows that

$$M_{\mathbb{C}}(r) \otimes M_{\mathbb{C}}(2) = M_{\mathbb{C}}(2r) \quad \forall\ r.$$

So to get (3.27) and (3.28) it suffices to observe that

$$\begin{aligned} \mathbb{C}\mathrm{l}(2) &\simeq M_{\mathbb{C}}(2) \\ \mathbb{C}\mathrm{l}(1) &\simeq \mathbb{C} \oplus \mathbb{C}. \end{aligned}$$

The first of these follows from the proof of Lemma 3.3. The second is given by the explicit isomorphism

$$\mathbb{C}\mathrm{l}(1) = \operatorname*{span}_{\mathbb{C}} \{1, \xi\} \longmapsto \operatorname*{span}_{\mathbb{C}} \left\{ \frac{1+\xi}{2}, \frac{1-\xi}{2} \right\}.$$

The proof of Lemma 3.3 also shows the importance of considering the $\mathbb{Z}_2$-*grading* of $\mathbb{C}\mathrm{l}(k)$ (or $\mathrm{Cl}(k)$)

$$\mathbb{C}\mathrm{l}(k) = \mathbb{C}\mathrm{l}^+(k) \oplus \mathbb{C}\mathrm{l}^-(k), \tag{3.29}$$

where $\mathbb{C}l^+(k)$ and $\mathbb{C}l^-(k)$ are respectively the sums of multiples of products of even and odd numbers of the generators. More generally

$$\mathrm{Cl}^+(V) = \bigoplus_{k=0}^{\infty} \mathrm{Ten}_{2k}(V) \Big/ \left(\mathcal{I}_{\mathrm{Cl}}(V) \cap \bigoplus_{k=0}^{\infty} \mathrm{Ten}_{2k}(V)\right)$$
$$\mathrm{Cl}^-(V) = \bigoplus_{k=0}^{\infty} \mathrm{Ten}_{2k+1}(V) \Big/ \left(\mathcal{I}_{\mathrm{Cl}}(V) \cap \bigoplus_{k=0}^{\infty} \mathrm{Ten}_{2k+1}(V)\right),$$

where (3.29) follows from the fact that $\mathcal{I}_{\mathrm{Cl}}$ is graded, i.e. it is the sum of its even and odd parts, since the relation in (3.10) respects parity. It is particularly important that (3.29) is a grading of the algebra:

$$\begin{aligned}
\mathrm{Cl}^+(V) \cdot \mathrm{Cl}^+(V) &\subset \mathrm{Cl}^+(V)\\
\mathrm{Cl}^-(V) \cdot \mathrm{Cl}^-(V) &\subset \mathrm{Cl}^+(V)\\
\mathrm{Cl}^+(V) \cdot \mathrm{Cl}^-(V) &\subset \mathrm{Cl}^-(V)\\
\mathrm{Cl}^-(V) \cdot \mathrm{Cl}^+(V) &\subset \mathrm{Cl}^-(V).
\end{aligned}$$

The $\mathbb{Z}_2$-grading defines, and is determined by, an involution, i.e. a linear map

$$R_{\mathrm{Cl}} \colon \mathrm{Cl}(V) \longleftrightarrow \mathrm{Cl}(V)$$

given by the obvious sign reversal:

$$R_{\mathrm{Cl}} \restriction \mathrm{Cl}^+(V) = \mathrm{Id},\ \ R_{\mathrm{Cl}} \restriction \mathrm{Cl}^-(V) = -\mathrm{Id}\,.$$

Next note how the relationship between the two cases (3.27), (3.28) can be understood directly. Suppose that V is a real vector space with a Euclidean inner product. On $V \times \mathbb{R}$ take the obvious extension of this

$$\langle (v,r), (v',r')\rangle = \langle v, v'\rangle + rr'.$$

Then the map, at the level of the tensor algebra

$$\begin{aligned}
\mathrm{Ten}_{\mathbb{C},2k}(V) \ni \alpha &\longmapsto \alpha \in \mathrm{Ten}_{\mathbb{C},2k}(V \times \mathbb{R})\\
\mathrm{Ten}_{\mathbb{C},2k+1}(V) \ni \alpha &\longmapsto i\alpha \otimes \tau \in \mathrm{Ten}_{\mathbb{C},2k+2}(V \times \mathbb{R}),
\end{aligned} \tag{3.30}$$

where $\tau = (0,1) \in V \times \mathbb{R}$, projects to an isomorphism

$$\mathbb{C}l(V) \longleftrightarrow \mathbb{C}l^+(V \times \mathbb{R}). \tag{3.31}$$

EXERCISE 3.4. Prove that (3.31) is an isomorphism of algebras.

3.4. Clifford bundle.

So far a general solution to (3.1)–(3.3) has not been obtained. Before doing so, consider how to transfer the notions above to a general Riemann manifold; eventually this will lead to the notion of a (generalized) Dirac operator. Certainly if X is a Riemann manifold then T^*_xX is, for each $x \in X$, a vector space with Euclidean inner product. So $\mathrm{Cl}_x(X) = \mathrm{Cl}(T^*_xX)$, $\mathbb{C}\mathrm{l}_x(X) = \mathbb{C}\mathrm{l}(T^*_xX)$ fixes the real and complex Clifford algebra at each point. These spaces combine into bundles of algebras over X

$$\mathrm{Cl}(X) = \bigsqcup_{x\in X} \mathrm{Cl}_x(X), \quad \mathbb{C}\mathrm{l}(X) = \bigsqcup_{x\in X} \mathbb{C}\mathrm{l}_x(X). \tag{3.32}$$

If $O \in \mathrm{O}(V)$ is an orthogonal transformation on V then O extends to $\mathrm{Ten}(V)$ as the sum of the tensor actions. The ideal $\mathcal{I}_{\mathrm{Cl}} \subset \mathrm{Ten}(V)$ is invariant under this action, since $\langle Ov, Ov'\rangle = \langle v, v'\rangle$ by definition. Thus $\mathrm{Cl}(V)$ has a natural $\mathrm{O}(V)$-action consistent both with the inclusion $V \subset \mathrm{Cl}(V)$ and the algebra structure, i.e.

$$O\alpha \cdot O\beta = O(\alpha \cdot \beta).$$

In this way $\mathrm{Cl}(X)$ is identified as a bundle associated to the orthonormal frame bundle:

$$\mathrm{Cl}(X) \simeq \big(F \times \mathrm{Cl}(N)\big)\big/\,\mathrm{O}(N),\ \dim X = N. \tag{3.33}$$

As discussed in Chapter 2 this means that $\mathrm{Cl}(X)$ has a natural Levi-Civita connection,

$$\nabla \in \mathrm{Diff}^1(X; \mathrm{Cl}(X), T^*X \otimes \mathrm{Cl}(X)).$$

Notice also that Clifford multiplication, defined by (3.18), is consistent with the $\mathrm{O}(V)$ action on Λ^*V, i.e.

$$\mathrm{cl}(O\xi) \cdot O\alpha = O(\mathrm{cl}(\xi) \cdot \alpha).$$

This means that the same formula, (3.18), extends to define Clifford multiplication on the manifold:

$$\mathcal{C}^\infty(X; \mathbb{C}\mathrm{l}(X)) \times \mathcal{C}^\infty(X; \Lambda^*X) \ni (\mu, \alpha) \longmapsto \mathrm{cl}(\mu) \cdot \alpha \in \mathcal{C}^\infty(X; \Lambda^*X).$$

This is a tensorial operation.

3.5. Clifford modules.

The exterior algebra, $\Lambda^* X$, is an example of a Clifford module over X. In general a vector bundle $E \longrightarrow X$ is a Clifford module[6] if there is a pointwise action, i.e. linear map

$$\mathrm{cl}\colon \mathrm{Cl}_x(X) \longrightarrow \hom(E_x) \tag{3.34}$$

such that the resulting tensorial operator is $\mathcal{C}^\infty$, i.e.

$$\mathcal{C}^\infty(X; \mathrm{Cl}(X)) \times \mathcal{C}^\infty(X;E) \longrightarrow \mathcal{C}^\infty(X;E),\ (\alpha, e) \longmapsto \mathrm{cl}(\alpha)e.$$

Usually the bundle E will be a complex bundle and the action that of the complexified Clifford algebra. By convention the Clifford action is assumed to be non-trivial.

By (3.27), in the $2k$-dimensional case, the complexified Clifford algebra is isomorphic to a full matrix algebra. It follows that for any (non-trivial) Clifford action (3.34) the subspace of invariant elements, $E'_x \subset E_x$, is of dimension $\dim E_x - 2^k$. Choosing a complementary space E''_x on which the algebra acts (simply by taking the span of the orbit of an element outside E'_x) gives the decomposition

$$E_x = E''_x \oplus E'_x \Longrightarrow \hom(E_x) = \mathbb{C}\mathrm{l}_x \otimes \hom'_{\mathbb{C}\mathrm{l}}(E_x), \tag{3.35}$$

where $\hom'_{\mathbb{C}\mathrm{l}}(E_x) \subset \hom(E_x)$ is the space of linear operators commuting with the Clifford action. Whilst the decomposition of the space is not unique, that of the homomorphism bundle is. It follows that the corresponding decomposition of the bundle $\hom(E)$ is smooth.

Suppose that E has a connection ∇. The obvious compatibility condition to demand for ∇ to be a 'Clifford connection,' is

$$\nabla_V(\mathrm{cl}(\alpha)e) = \mathrm{cl}(\nabla_V\alpha)e + \mathrm{cl}(\alpha)\nabla_V e,\ \forall\ V \in \mathcal{V}(X), \tag{3.36}$$

where $\nabla_V\alpha$ is the Levi-Civita connection on $\mathrm{Cl}(X)$.

EXERCISE 3.5. Show that any Clifford module has a Clifford connection. [Hint: See Lemma 3.24.]

LEMMA 3.6. *If* $\dim X$ *is even then the homomorphism bundle of any Clifford module over* X *decomposes smoothly as the tensor product*

$$\hom(E) = \mathbb{C}\mathrm{l}(X) \otimes \hom'_{\mathbb{C}\mathrm{l}}(E). \tag{3.37}$$

If E *has a Clifford connection then the curvature operator on* E *also decomposes as*

$$K_{\hom(E)} = K_{\mathbb{C}\mathrm{l}} \otimes \mathrm{Id} + \mathrm{Id} \otimes K_{\hom'_{\mathbb{C}\mathrm{l}}(E)}. \tag{3.38}$$

PROOF: The identity (3.37) has already been proved. Applying (2.76) to it gives (3.38).

[6] One should perhaps call these bundles of Clifford modules, but that is too clumsy.

DEFINITION 3.7. If $E \longrightarrow X$ is a Clifford module over a Riemann manifold X the generalized Dirac operator associated with a connection on E is

$$\eth_E = -i\tilde{c} \cdot \nabla \in \mathrm{Diff}^1(X; E), \tag{3.39}$$

given by composing the connection with $-i\tilde{c}$ where

$$\tilde{c} \colon \mathcal{C}^\infty(X; T^*X \otimes E) \longrightarrow \mathcal{C}^\infty(X; E),\ e \otimes \xi \longmapsto \mathrm{cl}(\xi) \cdot e. \tag{3.40}$$

Notice that by taking a local orthonormal coframe, ϕ^j, and the dual frame with respect to the metric, v_j, for TX the definition (3.39) becomes

$$\eth_E e = \sum_j \mathrm{cl}(\phi^j)\nabla_{iv_j}. \tag{3.41}$$

EXERCISE 3.8. Check directly that (3.41) is independent of the choice of orthonormal basis.

The pure imaginary factor is included in the definition so that $\eth_E$ is at least a symbolic square root of the Laplacian. More generally, if E is an Hermitian bundle, i.e. has a smoothly varying positive-definite sesquilinear inner product on the fibres, then a Clifford action is said to be Hermitian (or unitary) if

$$\mathrm{cl}(\xi)^* = \mathrm{cl}(\xi) \quad \forall\, \xi \in T^*X.$$

PROPOSITION 3.9. *If E is a Clifford module over X then the generalized Dirac operator satisfies*

$$\sigma_1(\eth_E)(\xi) = \mathrm{cl}(\xi) \quad \forall\, \xi \in T^*X. \tag{3.42}$$

If the bundle and Clifford action are Hermitian then

$$\sigma_2(\eth_E^*\eth_E)(\xi) = |\xi|^2 \quad \forall\, \xi \in T^*X \tag{3.43}$$

and if the connection is unitary and a Clifford connection then $\eth_E$ is self-adjoint.

PROOF: The symbol map is a homomorphism so the symbol of $\eth_E$ is given by the composite of the symbol of the connection, which is the tensor product with $i\xi$, and $-i\tilde{c}$, i.e. at $\xi \in T_x^*X$

$$E_x \ni e \longmapsto e \otimes \xi \longmapsto \mathrm{cl}(\xi) \cdot e$$

which gives (3.42). If E is Hermitian then the adjoint of $\eth_E$, using the Riemann density on X, is well defined and has symbol given by the adjoint of the symbol, i.e. $\mathrm{cl}(\xi)^*$. If the Clifford action is Hermitian this reduces to $\mathrm{cl}(\xi)$ and hence (3.43) follows from the multiplicativity of the symbol map and the definition of the Clifford algebra.

The inner product on E induces a natural inner product on $T^*X \otimes E$ given by the usual tensor product formula

$$\langle \xi \otimes e, \eta \otimes f\rangle = \langle \xi, \eta\rangle\langle e, f\rangle.$$

This has the property that if $\xi \in \mathcal{C}^\infty(X; T^*X)$ is real and $V \in \mathcal{V}(X)$ is the dual vector field then

$$\langle \nabla e, \xi \otimes f\rangle = \langle \nabla_V e, f\rangle.$$

If the connection on E is unitary, i.e. (2.108) holds, then it follows that the adjoint of the connection is

$$\nabla^*(\xi \otimes e) = -\nabla_V e + (\operatorname{div} V)e. \tag{3.44}$$

Here $\operatorname{div} V$ is the metric divergence of V defined as the difference of the adjoint of $V \in \mathrm{Diff}^1(X)$ and $-V$. Similarly if the Clifford action is Hermitian then the adjoint of (3.40) satisfies

$$\langle \tilde{c}^* e, \xi \otimes f\rangle = \langle e, \mathrm{cl}(\xi)\cdot f\rangle = \langle \mathrm{cl}(\xi)\cdot e, f\rangle.$$

Thus $\tilde{c}^*(e)(v_j) = \mathrm{cl}(\phi^j)e$ if ϕ^j and v_j are dual local orthonormal bases of T^*X and TX respectively, i.e.

$$\tilde{c}^*(e) = \sum_i \phi^j \otimes \mathrm{cl}(\phi^j)e.$$

Combining this with (3.44), and using (3.36), this shows that with respect to such a basis

$$\eth_E^* e = i\sum_j \nabla^*_{v_j}[\mathrm{cl}(\phi^j)\cdot e]$$

$$= -i\sum_j \mathrm{cl}(\phi^j)\nabla_{v_j}\cdot e + i\sum_j \mathrm{div}(v_j)\,\mathrm{cl}(\phi^j)\cdot e + ic\left(\sum_j \nabla_{v_j}\phi^j\right)e.$$

The first sum on the right is just $\eth_E e$. The remaining part vanishes since it must be independent of the choice of orthonormal basis and the basis can be chosen covariant constant at any given point, in which case both $\operatorname{div}(v_j)$ and $\nabla_{v_j}\phi^j$ vanish at that point. Thus $\eth_E$ is self-adjoint as claimed.

Particularly in the even-dimensional case it is natural to consider Clifford modules which are graded. Thus the bundle has a decomposition

$$E = E^+ \oplus E^-$$

such that

$$\mathbb{C}\mathrm{l}^+_x : E^\pm_x \longrightarrow E^\pm_x,\ \mathbb{C}\mathrm{l}^-_x : E^\mp_x \longrightarrow E^\pm_x\ \forall\ x \in X.$$

From the formula (3.41) and Proposition 3.9 one easily deduces:

COROLLARY. *If X is an even-dimensional Riemann manifold and E is a graded Hermitian Clifford module with a connection which is unitary, Clifford and preserves the grading then the Dirac operator is graded in the sense that*

$$\eth_E = \begin{pmatrix} 0 & \eth^-_E \\ \eth^+_E & 0 \end{pmatrix},\ (\eth^+_E)^* = \eth^-_E.$$

3.6. Clifford bundle of ${}^bTX$.

The definitions above extend directly to the case of a b-metric. Thus (3.32) and (3.33) serve just as well to define the Clifford bundle, over any manifold with boundary, associated to a b-metric, now of course with ${}^b\mathrm{Cl}_x(X) = \mathrm{Cl}({}^bT^*_xX)$. The Levi-Civita connection of the b-metric extends to a b-connection on ${}^b\mathrm{Cl}(X)$. If the metric is an exact b-metric then Proposition 2.37 shows that the b-connection is actually a connection on ${}^bT^*X$, and hence on $\mathrm{Cl}({}^bT^*X)$. Simply by continuity the consistency condition demanded of a b-Clifford connection should be

$${}^b\nabla(\mathrm{cl}(\alpha)\cdot e) = \mathrm{cl}({}^b\nabla\alpha)\cdot e + \mathrm{cl}(\alpha)\cdot {}^b\nabla e.$$

Then the proof of Proposition 3.9 applies, essentially *verbatim,* to give the b-analogue:

PROPOSITION 3.10. *If E is a Clifford module for an exact b-metric on a compact manifold with boundary, X, the generalized Dirac operator ${}^b\eth_E \in \mathrm{Diff}^1_b(X;E)$ satisfies ${}^b\sigma_1({}^b\eth_E)(\xi) = \mathrm{cl}(\xi)$ for $\xi \in {}^bT^*X$ and if the bundle and Clifford action are Hermitian then ${}^b\sigma_2({}^b\eth^*_E\,{}^b\eth_E) = |\xi|^2$ for $\xi \in {}^bT^*X$, with ${}^b\eth_E$ self-adjoint for a unitary Clifford connection.*

EXERCISE 3.11. Go through the proof of this Proposition carefully to make sure that you understand why this generalization to the b-metrics is *immediate!*

Certainly the Corollary to Proposition 3.9 also extends by continuity to the case of a generalized Dirac operator associated to a graded Hermitian Clifford module over an even-dimensional exact b-manifold when the connection on the module is graded, Clifford and unitary. It is not necessary to assume that the connection is a true connection, only that it is a b-connection. This is especially convenient because it allows the weighting discussed in §6 to be incorporated in the general case. The assumption that the metric is an exact b-metric and that the connection is Clifford means that the part of the connection on the Clifford factor in (3.37) will always be a true connection.

3.7. Spin group.

Naturally the question arises as to the existence of Clifford modules. One way these arise is through the notion of a spin structure and the associated spinor bundles. This leads to the definition of 'the' Dirac operator and more generally the twisted Dirac operators which appear in the APS theorem of the introduction. They were introduced in this degree of generality by Atiyah and Singer [13].

The first thing to consider is the relationship between the Clifford algebra and the orthogonal group of a Euclidean vector space. Recall that there is a natural embedding $V \hookrightarrow \mathrm{Cl}(V)$. Consider the elements of V of unit length as elements of $\mathrm{Cl}(V)$. The anticommutation relation

$$\xi \cdot \eta + \eta \cdot \xi = 2\langle \xi, \eta\rangle \,\mathrm{Id}, \quad \xi, \eta \in V$$

can then be rewritten

$$\xi \cdot \eta - \big(-\eta + 2\langle \xi, \eta\rangle \xi\big) \cdot \xi = 0, \ |\xi| = 1, \tag{3.45}$$

since $\xi^2 = \mathrm{Id}$. Notice that

$$\hat{\eta} = \eta - 2\langle \xi, \eta\rangle\xi = R_\xi \eta$$

is the reflection of η in the plane orthogonal to ξ. Thus for any $\xi \in V, \ |\xi| = 1$

$$R_\xi v = v - 2\langle \xi, v\rangle \xi \quad \forall\ v \in V \tag{3.46}$$

is an element of $\mathrm{O}(V)$.

The anticommutation relations amongst these orthogonal involutions are:

$$R_\xi \cdot R_\eta = R_{\hat{\eta}} \cdot R_\xi \ , \ \hat{\eta} = R_\xi \eta, \ |\xi| = |\eta| = 1 \tag{3.47}$$

as follows directly from (3.46). Since these are the same relations as (3.45) (noting that $R_\xi = R_{-\xi}$) and are the only relations amongst these involutions, it follows that the map

$$\{\eta \in V \subset \mathrm{Cl}(V); \ |\eta| = 1\} \longrightarrow \mathrm{O}(V) \tag{3.48}$$

can be extended multiplicatively. That is, setting

$$\mathrm{Pin}(V) = \{a \in \mathrm{Cl}(V); a = \eta_1 \cdot \dots \cdot \eta_p \text{ for some } p, \eta_i \in V, \ |\eta_i| = 1\}$$

leads to

PROPOSITION 3.12. *The map generated by* (3.48) *is a group homomorphism*

$$Z\colon \operatorname{Pin}(V) \longrightarrow \mathrm{O}(V) \tag{3.49}$$

which is a 2-1 *covering of the orthogonal group, with kernel* $\pm\operatorname{Id} \in \mathrm{Cl}(V)$.

PROOF: First observe that $\operatorname{Pin}(V)$ *is* a group, since it has an associative product and the inverse of $\eta_1 \dots \eta_k$ is just $\eta_k \dots \eta_1$. Consider (3.48) another way. Namely the orthogonal transformation R_η can be written

$$V \ni v \longmapsto R_\eta v = -\eta \cdot v \cdot \eta, \quad |\eta| = 1,\ \eta \in V$$

in terms of Clifford multiplication. Indeed if $v = a\eta + b\eta^\perp$, $\langle \eta, \eta^\perp \rangle = 0$, then

$$\eta \cdot v = a - b\eta^\perp \cdot \eta \Longrightarrow \eta \cdot v \cdot \eta = a\eta - b\eta^\perp = -R_\eta v. \tag{3.50}$$

This shows directly that (3.49), given by (3.46), is a group homomorphism.

It remains to show that the map Z is surjective and to find its kernel. Any orthogonal transformation can be written as a product of reflections, in fact a product of at most $\dim V$ reflections.

EXERCISE 3.13. Recall how to prove this. If $O \in \mathrm{O}(V)$ then O is unitary (being real $O^* = O^t = O^{-1}$) so has a spectral decomposition

$$O = \sum_j \lambda_j P_j,$$

where the P_j are complex projections onto the eigenspaces associated to the λ_j. The λ_j's all have modulus 1 and, since O is real, come in complex-conjugate pairs, or are real, and similarly for the P_j's. This decomposes V into a sum of one and two dimensional orthogonal spaces

$$V = \bigoplus_j V_j$$

which O preserves. This reduces the problem to the one and two dimensional cases, where it is easy enough.

If $O = R_{\eta_1} \cdot \dots \cdot R_{\eta_k}$, with $|\eta_i| = 1$, $\eta_i \in V$, then $O = Z\big(\pm\eta_1 \dots \eta_k\big)$ as is clear from (3.50). This shows the surjectivity.

Clearly $Z(\pm\operatorname{Id}) = \operatorname{Id}$. Conversely suppose $Z(a) = \operatorname{Id}$, $a \in \operatorname{Pin}(V)$. From (3.50) this means that if $a = \eta_1 \dots \eta_k \in \mathrm{Cl}^{(k)}(V)$

$$\eta_1 \cdot \dots \cdot \eta_k v = (-1)^k v\eta_1 \dots \eta_k \quad \forall\ v \in V.$$

Thus a must satisfy

$$a \cdot v = (-1)^k v \cdot a \quad a \in \mathrm{Cl}^k(V). \tag{3.51}$$

Now decompose a into its unique presentation as a sum over the basis of the subspace of $\mathrm{Cl}^{(k)}(V)$ of the same parity as k, given by products $e_{i_1} \dots e_{i_l}$ $i_1 < i_2 < \cdots < i_l$, $l \equiv k$ modulo 2, where the $e_i \in V$ form an orthonormal basis:

$$a = \sum_{l \equiv k \mod 2} c_I e_{i_1} \dots e_{i_l}.$$

Applying (3.51) for each j, $v = e_j$, shows that $c_I = 0$ if $i_r = j$ for some r. Thus $a = c_0 \,\mathrm{Id}$ and then (3.51) can hold if and only if $c_0 = \pm 1$, $k = 0$. This proves the proposition.

The orthogonal group has two components:

$$\mathrm{O}(V) = \mathrm{SO}(V) \sqcup \mathrm{O}^-(V), \ \dim V \geq 2,$$

where

$$O \in \mathrm{SO}(V) \Longleftrightarrow O \in \mathrm{O}(V) \text{ and } \det(O) = 1.$$

Since a reflection has determinant -1, $O \in \mathrm{SO}(V)$ if and only if it is a product of an *even* number of reflections. Thus, in (3.47),

$$\mathrm{Spin}(V) \stackrel{\text{def}}{=} Z^{-1}(\mathrm{SO}(V)) = \mathrm{Pin}(V) \cap \mathrm{Cl}^+(V).$$

LEMMA 3.14. *If* $\dim V \geq 3$, *the group* $\mathrm{Spin}(V) = \mathrm{Pin}(V) \cap \mathrm{Cl}^+(V)$ *is the unique simply-connected double cover of* $\mathrm{SO}(V)$.

PROOF: Since $\mathrm{SO}(V)$ is connected and Z is a double cover, $\mathrm{Spin}(V)$ can have at most two components.

EXERCISE 3.15. Show that $\mathrm{SO}(V)$ is connected. Choose an orthonormal basis of V so that the problem is reduced to $V = \mathbb{R}^N$, $\mathrm{SO}(V) = \mathrm{SO}(N)$. Given $O \in \mathrm{SO}(N)$ let $f^j = O(e^j)$ be the image of the standard orthonormal basis. It is easy to check that $\mathrm{SO}(2)$ is connected. Using a rotation in the first two variables it can be arranged that $f^2 \perp e^1$. Similarly a rotation in the first and third variables allows one to arrange $f^3 \perp e^1$. Thus O is connected in $\mathrm{SO}(N)$ to an element O' such that the corresponding basis $(f')^j \perp e^1$, $j \geq 2$. This means $f^1 = \pm e^1$, so by a further rotation in the first two variables it can be assumed that $f^1 = e^1$. Now proceed by induction.

To show that $\mathrm{Spin}(V)$ is connected it therefore suffices to check that -1 is connected to 1. If $\xi \perp \eta$, with $|\xi| = |\eta| = 1$, then

$$\xi\eta\xi\eta = -\xi\xi\eta\eta = -1.$$

Since, for $\dim V \geq 2$, the sphere is connected, a path ξ_t in V with $|\xi_t| = 1$, $\xi_0 = \eta$, $\xi_1 = \xi$, can be chosen and then $\xi_t\eta\xi_t\eta$ connects 1 to -1. In fact a similar argument shows that if $\dim V \geq 3$ then $\mathrm{Spin}(V)$ is simply connected. In this case the unit sphere in V is simply connected. Any curve in $\mathrm{Spin}(V)$ can be approximated closely by a curve of the form $\xi_1(t) \cdot \xi_2(t) \cdot \ldots \cdot \xi_N(t)$, where $N = \dim V$ or $\dim V - 1$, as $\dim V$ is even or odd. Each of these $\xi_j(t)$ are curves on the unit sphere so can be contracted to a fixed element. Since $\mathrm{SO}(N)$ is connected this shows that the double cover

$$Z\colon \mathrm{Spin}(V) \longrightarrow \mathrm{SO}(V) \tag{3.52}$$

is non-trivial, and in fact yields the well-known result that the fundamental group of $\mathrm{SO}(V)$ is $\mathbb{Z}_2$, if $\dim V \geq 3$.

The spin group of the Euclidean vector space V has been defined as a submanifold of the vector space $\mathrm{Cl}^+(V)$, so the Lie algebra must be a linear subspace. In fact

LEMMA 3.16. *The Lie algebra of* $\mathrm{Spin}(V) \subset \mathrm{Cl}^+(V)$ *is the linear span of antisymmetric products:*

$$\mathfrak{sn}(V) = \mathrm{span}\,\{\xi\xi' - \xi'\xi; \xi, \xi' \in V\} \subset \mathrm{Cl}^+(V). \tag{3.53}$$

PROOF: Let $\xi \in V$ be a unit vector and choose $\xi' \perp \xi$ with $|\xi'| = 1$. Then setting

$$\xi_s = (1-s^2)^{\frac{1}{2}}\xi + s\xi' \in V,\ |\xi_s| = 1 \Longrightarrow c(s) = \xi\xi_s \in \mathrm{Spin}(V). \tag{3.54}$$

Moreover $\xi_0 = \xi$ so $c(0) = \mathrm{Id}$. Thus the tangent vector at $s = 0$ to the curve $c(s)$ is an element of $\mathfrak{sn}(V)$, it is just

$$c'(0) = \xi\xi' = \frac{1}{2}(\xi\xi' - \xi'\xi) \text{ since } \xi \perp \xi'.$$

Taking linear combinations, it follows that the subset of the space on the right in (3.53) with $\xi \perp \xi'$ is contained in $\mathfrak{sn}(V)$. Since $\xi\xi = 1$, $\mathfrak{sn}(V)$ contains the vector space in (3.53). The dimensions being the same, this proves the lemma.

The double covering (3.52) gives a canonical identification of the Lie algebras, $\mathfrak{so}(N)$ of $\mathrm{SO}(N)$ and $\mathfrak{sn}(N)$ of $\mathrm{Spin}(N)$. Since the former is also naturally identified with the linear space of antisymmetric real matrices, (3.53) gives a natural map

$$\begin{aligned}\mathfrak{so}(n) = \{A_{ij}; A_{ij} + A_{ji} = 0,\ i,j = 1,\dots,n\} \longrightarrow \\ \operatorname{span}\{\xi\xi' - \xi'\xi; \xi,\xi' \in V\} = \mathfrak{sn}(n) \subset \mathrm{Cl}^+(n).\end{aligned} \tag{3.55}$$

Lemma 3.17. *The natural identification* (3.55) *is*

$$A_{ij} \longmapsto \frac{1}{4}\sum_{i,j=1}^{n} A_{ij}\phi^i\phi^j. \tag{3.56}$$

Proof: Consider the curve (3.54) in $\mathrm{Spin}(N)$ with $\xi = \phi^i$ and $\xi' = \phi^j$. It has tangent vector $\frac{1}{2}(\phi^i\phi^j - \phi^j\phi^i)$ at 0. By (3.46) and (3.47) its image as an orthogonal transformation is

$$v \longmapsto v - 2\langle v, e_s\rangle e_s - 2\langle v, e^i\rangle e^i + 4\langle v, e_s\rangle\langle e_s, e^i\rangle e^i,\ \ e_s = (1-s^2)^{\frac{1}{2}}\phi^i + s\phi^j.$$

The tangent vector at $s = 0$ to this curve in $\mathrm{SO}(N)$ is just

$$v \longmapsto -2v_i\phi^j + 2v_j\phi^i.$$

This is the correspondence (3.56).

3.8. Spin representations.

Next the embedding $\mathrm{Spin}(V) \hookrightarrow \mathrm{Cl}(V)$ will be used to find the spin representations, which are representations of $\mathrm{Spin}(V)$ but *not* representations of $\mathrm{SO}(V)$. In fact (3.26) already gives a representation of $\mathbb{C}\mathrm{l}(V)$. Thus, if $\dim V = 2k$, there is an identification $\mathrm{Cl}(V) \simeq \mathrm{Cl}(2k)$ and then

$$c\colon\ \mathbb{C}\mathrm{l}(2k) \longleftrightarrow M_{\mathbb{C}}(2^k). \tag{3.57}$$

This gives a representation

$$\mathrm{Pin}(V) \longrightarrow GL(2^k,\mathbb{C}),\quad \dim V = 2k. \tag{3.58}$$

Moreover (3.58) must be irreducible, since the $\mathbb{C}$-linear span of $\mathrm{Pin}(V)$ is $\mathbb{C}\mathrm{l}(V)$ which certainly has no invariant subspace (by (3.57).)

So consider what happens when (3.58) is restricted to the connected component of the identity, i.e. $\mathrm{Spin}(V)$.

LEMMA 3.18. *If* $\dim V = 2k$, *the restriction of* (3.58) *to the spin subgroup:*

$$\text{SR}\colon \operatorname{Spin}(V) \longrightarrow U(2^k) \tag{3.59}$$

is a unitary representation which is the direct sum of two irreducible representations

$$\text{SR}^{\pm}\colon \operatorname{Spin}(V) \longrightarrow GL({}^{\pm}S), \quad \mathbb{C}^{2^k} = S = {}^{+}S \oplus {}^{-}S. \tag{3.60}$$

PROOF: From (3.31) it follows that $\mathbb{Cl}^{+}(2k) \cong \mathbb{Cl}(2k-1)$. Then (3.28) shows that this is isomorphic to two copies of $M_{\mathbb{C}}(2^{k-1})$. The restriction of the representation (3.58) to $\mathrm{Cl}^{+}(2k)$ therefore decomposes into two representations, where $\dim {}^{+}S = \dim {}^{-}S = 2^{k-1}$. Since the $\mathbb{C}$-linear span of $\operatorname{Spin}(2k)$ is $\mathbb{Cl}^{+}(2k)$, it follows that these two representations are irreducible and (3.60) holds.

Notice that the combined representation is in fact unitary. This can be seen directly from the discussion leading to (3.28). In $\mathbb{Cl}(2k)$ there is a natural involution, defined as the identity on V and 1 and extended to be conjugate-linear and product reversing:

$$\alpha^* = \sum_I c_I e_{i_1} \cdots e_{i_j} \Longrightarrow \alpha^* = \sum_I \bar{c}_I e_{i_j} \cdots e_{i_1}.$$

From (3.27) and (3.28) it follows that the isomorphism (3.24) is a $*$-isomorphism, with the usual adjoint operation on $M_{\mathbb{C}}(2)$. Since the elements of $\operatorname{Spin}(V)$ are unitary, in the sense that $S^* = S^{-1}$ if $S \in \operatorname{Spin}(V)$, the representation (3.59) is unitary.

The odd-dimensional case is even simpler.

LEMMA 3.19. *If* $V = \mathbb{R}^{2k+1}$ *the restriction of the representation*

$$\mathbb{Cl}^{+}(V) \longleftrightarrow \mathbb{Cl}(2k) \longleftrightarrow \hom(\mathbb{C}^{2^k}) \tag{3.61}$$

to $\operatorname{Spin}(V)$ *is an irreducible unitary representation.*

PROOF: The first identification in (3.61), given by (3.31), is a $*$-isomorphism, so restriction to $\operatorname{Spin}(V)$ gives a unitary representation. Since $\operatorname{Spin}(V)$ spans $\mathbb{Cl}^{+}(V)$, which is to say $M_{\mathbb{C}}(2^k)$, over $\mathbb{C}$ the representation is necessarily irreducible.

Alternatively the spin representation in the odd dimensional case, $V = \mathbb{R}^{2k+1}$, can be obtained from (3.31) as the identification

$$\mathbb{Cl}(V) \longleftrightarrow \mathbb{Cl}^{+}(2k+2),$$

allowing either of the spin representations of $\mathrm{Spin}(2k+2)$ to be restricted to $\mathrm{Spin}(V)$ where they give equivalent irreducible representations, also equivalent to that above. This is discussed in more detail in §3.13 below.

These two lemmas fix the spin representations of $\mathrm{Spin}(\mathbb{R}^n)$ and solve the problem posed at the beginning of this chapter. Namely if $n = 2k$ the $2^{k-1} \times 2^{k-1}$ matrices

$$\gamma_j = i\,\mathrm{SR}^+(\phi^j\phi^1),\ j = 1, \dots, 2k$$

defined by the representation satisfy (3.4), with $\gamma_1 = i\,\mathrm{Id}$. This gives $\eth^+$ defined by (3.1) satisfying (3.2). In fact, just taking the γ_i to be $\mathrm{SR}^+(\phi^i)$ gives a solution of (3.3). As the notation indicates the representation of the Clifford algebra in (3.57) is generally written as Clifford multiplication:

$$\mathrm{cl}(\alpha) \in \hom(S),\ \xi \in \mathbb{C}l(2k).$$

Then the full Dirac operator can be written

$$\eth = \sum_{j=1}^{2k} \mathrm{cl}(\phi^j) D_j.$$

EXERCISE 3.20. Check that $\eth^2 = \Delta\,\mathrm{Id}$.

3.9. Spin structures.

To transfer this construction to a Riemann manifold requires the introduction of the notion of a spin structure. As discussed in §2.12 an orientation on X corresponds to a reduction of the structure group of the coframe bundle to $\mathrm{SO}(N)$. Similarly one can ask when the bundle of oriented orthonormal frames can be extended to a principal $\mathrm{Spin}(N)$-bundle.

DEFINITION 3.21. A principal $\mathrm{Spin}(N)$-bundle which is a double cover $s\colon {}_sF \longrightarrow {}_oF$ is called a *spin structure* if the $\mathrm{Spin}(N)$ action is consistent with the $\mathrm{SO}(N)$ action, i.e. if $A \in \mathrm{Spin}(N)$ and $\pi(A) \in \mathrm{SO}(N)$ is its image then the diagram

$$\begin{tikzcd} {}_sF_x \arrow[r, "A"] \arrow[d, "s"'] & {}_sF_x \arrow[d, "s"] \\ {}_oF_x \arrow[r, "\pi(A)"'] & {}_oF_x \end{tikzcd} \tag{3.62}$$

commutes.

EXERCISE 3.22. Recall, for example from [67], the definition of the Stiefel-Whitney classes $w_i \in H^i(X, \mathbb{Z}_2)$ of (the tangent bundle of) an oriented manifold. Show that the existence of a spin structure on X is equivalent to the vanishing of w_2.

The fact that the covering map, $s\colon {}_sF \longrightarrow {}_oF$, is discrete means that the Levi-Civita connection on F, which certainly restricts to ${}_oF$ when the manifold is oriented, lifts to a unique, compatible, connection on ${}_sF$. The compatibility condition is just the requirement that the image of the connection on ${}_sF$ be that on ${}_oF$ under s_*.

The existence of a spin structure on X allows the spin bundles to be defined as bundles associated to the principal $\mathrm{Spin}(N)$-bundle ${}_sF$. Thus the fibre at any point $x \in X$ is just

$$S_x = ({}_sF_x \times S(N)) \,/\, \mathrm{SR}, \;\; N = \dim X, \tag{3.63}$$

where $\mathrm{SR}\colon \mathrm{Spin}(N) \longrightarrow \hom(S(N))$ is the spin representation on $S(N)$. That is,

$$S = ({}_sF \times S(N)) \,/\, \mathrm{SR}$$

is a complex vector bundle over X of dimension 2^k where $n = 2k$ or $n = 2k+1$. In the even dimensional case the splitting of the spin representation gives a global splitting of the bundle:

$$S = {}^+S \oplus {}^-S, \;\; \dim X = 2k. \tag{3.64}$$

Since the spin representations are unitary the spin bundles are Hermitian, i.e have a natural positive definite Hermitian form on their fibres.

The Levi-Civita connection on ${}_sF(X)$ induces a connection on the spin bundles, and this connection is unitary. It is also clear that the spin bundles are Clifford modules. Certainly the spin group $\mathrm{Spin}(T^*_xX)$ acts on each fibre S_x and by definition the spin representation extends to a representation of $\mathrm{Pin}(T^*_xX)$ and to a representation of $\mathbb{C}\mathrm{l}(T^*_xX)$. Indeed this is how the spin representations were defined in the first place. It is important to notice for later reference that, again directly from the definition, in the even-dimensional case the action of $\mathbb{C}\mathrm{l}(T^*_xX)$ on S_x actually gives all linear transformations.

LEMMA 3.23. *If X is an even-dimensional spin manifold then*

$$\mathbb{C}\mathrm{l}(T^*_xX) \longleftrightarrow \hom(S_x) \quad \forall\; x \in X \tag{3.65}$$

gives an isomorphism of the Clifford bundle and the homomorphism bundle of the spinor bundle.

This is a special case of Lemma 3.6.

3.10. Clifford connections.

The spin bundle of a manifold with spin structure is an Hermitian Clifford module. To see that the Levi-Civita connection on it is a unitary Clifford connection it remains only to show that (3.36) holds. This follows directly from the definitions, since if $\alpha \in \mathcal{C}^\infty(X;\Lambda^1)$ and $s \in \mathcal{C}^\infty(X;S)$ are both covariant constant at a point p then so is $\mathrm{cl}(\alpha)s$. Using these examples we can now easily prove:

Lemma 3.24. *Any Clifford module over an even-dimensional manifold has a Clifford connection and any Hermitian Clifford module has a unitary Clifford connection.*

Proof: Suppose first that X is even dimensional. Lemma 3.6 and (3.65) can be used to decompose the given Clifford bundle as a tensor product $S \otimes G$ locally near any point, with S the spin bundle for a spin structure near that point. This shows that there is always a Clifford connection locally; superposition using a partition of unity gives a globally defined Clifford connection. If the Clifford bundle is Hermitian this also allows the connection to be chosen unitary.

Exercise 3.25. Extend Lemma 3.24 to the odd-dimensional case.

For later computations it is also useful to have an explicit formula for covariant differentiation of sections of the spin bundle, analogous to (2.66) for 1-forms and its extension to higher forms in Exercise 2.26. Let ϕ^i be an oriented local orthonormal coframe, i.e. a local orthonormal basis of T^*X which defines a local section of ${}_oF$, and let v_i be the dual orthonormal basis of TX near some point p. Then define, by analogy with (2.67), the Christoffel symbols for this frame by

$$\nabla_{v_i}\phi^j = -\sum_{k=1}^{N} \gamma^j_{ik}\phi^k. \tag{3.66}$$

Suppose f^i is a local section of ${}_sF$ which covers ϕ^i; there are two possible choices by (3.62). Then a section $s \in \mathcal{C}^\infty(X;S)$ of S, the spinor bundle, is locally reduced to map a s_f into $S(N)$, the spinor space for $\mathbb{R}^N$ by definition in (3.63).

Lemma 3.26. *With this notation for a section s of the spinor bundle*

$$(\nabla_{v_i}s)_f = v_i s_f(x) - \frac{1}{4}\Gamma_i(x)s,\ \ \Gamma_i(x) = \sum_{k,j=1}^{N} \gamma^k_{ij}\phi^j\phi^k, \tag{3.67}$$

where $\Gamma_i(x) \in \mathrm{Cl}_x$ acts by Clifford multiplication on the fibres of S.

PROOF: It suffices to prove (3.67) at each point p. If the basis ϕ^i happens to be covariant constant at p then the Christoffel symbols vanish at that point, the spin frame f^i is also covariant constant and so (3.67) holds at p. In general a frame ψ^i equal to ϕ^i at p and covariant constant at p can always be found. Indeed

$$\psi^i = \sum_{j=1}^{n} O_{ij}(x)\phi^i, \; O_{ij} = \delta_{ij} + A_{ij}(x),$$

where $A_{ij}(p) = 0$. The associated frame of ${}_sF$

$$g^i = \sum_{j=1}^{n} O_{ij}(x) f^j \text{ is covariant constant at } p.$$

Thus if $s_{f'}$ is the map into $S(N)$ representing s with respect to this frame then

$$(\nabla_{v_i} s)_{f'} = v_i s_{f'}.$$

Since $O_{ij}(p) = \delta_{ij}$

$$\begin{aligned}(\nabla_{v_i} a)_f(p) =& (\nabla_{v_i} s)_{f'}(p) = v_i s_{f'}(p) \\ =& v_i \left(\mathrm{SR}(O(x)) s_f\right)(p) = v_i s_f + \frac{1}{4} G^i s_f.\end{aligned}$$

Here Lemma 3.17 is used to write the action of the Lie algebra, so

$$G = \sum_{pq} G_{pq} \phi^p \phi^q$$

is acting by Clifford multiplication, where $G^i_{pq} = v_i A_{pq}$. To find the G^i_{ij} in terms of the Christoffel symbols, simply use the same computation for the connection on T^*X :

$$\nabla_{v_i} \xi^j(p) = \nabla_{v_i} (\sum_{k=1}^{n} O_{jk} \psi^k(p) = \sum_{k=1}^{n} (v_i O_{jk}) \phi^k(p) = \sum_k v_i A_{jk} \phi^k.$$

Comparing this with (3.66) gives the formula (3.67).

As a direct consequence of this computation we can deduce a formula for the curvature of the Levi-Civita connection on the spin bundle:

LEMMA 3.27. *For the Levi-Civita connection on the spin bundle on a spin manifold*

$$K_S(v,w) = -\frac{1}{4}\operatorname{cl} R(v,w). \tag{3.68}$$

3.11. Twisted Dirac operators.

Definition 3.7 fixes the Dirac operator $\eth \in \operatorname{Diff}^1(X;S)$ whenever X is a spin manifold, i.e. an oriented Riemann manifold with spin structure. This operator is often considered to be *the* Dirac operator. More generally suppose that E is any other $\mathcal{C}^\infty$ Hermitian bundle over X with a unitary connection. Then on the bundle $S \otimes E$ there is both a unitary connection and a Clifford action (acting purely on S). Again using Definition 3.7 this gives the 'twisted Dirac operator' $\eth_E \in \operatorname{Diff}^1(X; S \otimes E)$.

PROPOSITION 3.28. *The twisted Dirac operator on a compact spin manifold (without boundary), X, for an Hermitian coefficient bundle, E, with unitary connection is a formally self-adjoint operator which, if the dimension is even, decomposes into a sum:*

$$\eth_E = \begin{pmatrix} 0 & \eth_E^- \\ \eth_E^+ & 0 \end{pmatrix},\ \dim X = 2k, \tag{3.69}$$

where

$$\eth_E^\pm \in \operatorname{Diff}^1(X; {}^\pm S \otimes E, {}^\mp S \otimes E).$$

PROOF: The self-adjointness is a special case of Proposition 3.9. If $\dim X$ is even then the connection restricts to ${}^\pm S$ since both are associated bundles to the spin bundle. The connection is therefore graded and the Corollary to Proposition 3.9 applies.

3.12. Spin structure for a b-metric.

Finally having defined the notion of a twisted Dirac operator on a spin manifold consider the case of a b-spin manifold. Thus let X be a compact manifold with a b-metric. Again the orientability of X is equivalent to the existence of a subbundle

$${}^b_oF \subset {}^bF, \text{ with structure group } \mathrm{SO}(N).$$

Further consider the existence of a b-spin structure, meaning a principal $\mathrm{Spin}(N)$ bundle which double covers ${}^b_oF$ and is consistent with it, as in (3.62).

EXERCISE 3.29. Show that a b-spin structure exists if and only if a spin structure exists in the ordinary sense.

The Levi-Civita connection, a b-connection, lifts to a connection on the spin frame bundle which will be denoted ${}^b_sF$. Of course over the interior of X these notions just reduce to those discussed above. Thus the Dirac operator, or more generally the twisted Dirac operator for a coefficient bundle E with b-connection, is defined as a differential operator over the interior of X. More precisely

LEMMA 3.30. *The twisted Dirac operator $\eth_E$ for a bundle with b-connection over a b-spin manifold is an element of* $\mathrm{Diff}^1_b(X; S \otimes E)$.

PROOF: As in Definition 3.7, $i{}^b\eth_E$ is just the composite of the b-connection and Clifford multiplication:

$$
\begin{aligned}
{}^b\nabla : \mathcal{C}^\infty(X;E) \longrightarrow \mathcal{C}^\infty(X; {}^bT^*X \otimes E),\ {}^b\nabla \in \mathrm{Diff}^1_b(X; E, {}^bT^*X \otimes E)\\
\tilde{c} : \mathcal{C}^\infty(X; {}^bT^*X \otimes E) \longrightarrow \mathcal{C}^\infty(X;E),\ e \otimes \xi \longmapsto \mathrm{cl}(\xi) \cdot e.
\end{aligned}
$$

EXERCISE 3.31. Show that for any Hermitian bundle, E, with unitary b-connection over a b-spin manifold X (compact with boundary) the twisted Dirac operator defines a formally self-adjoint element ${}^b\eth_E \in \mathrm{Diff}^1_b(X; S\otimes E)$ with a splitting (3.69) if the dimension is even.

The general case considered in Chapter 9 is covered by the following consequence of this discussion:

LEMMA 3.32. *Let E be an Hermitian Clifford bundle over a compact manifold with boundary with an exact b-metric. Suppose further that E has a b-connection which is Hermitian and Clifford. Then the Dirac operator in the sense of Definition 3.7 is a self-adjoint element of* $\mathrm{Diff}^1_b(X;E)$. *If $E = E^+ \oplus E^-$ is graded, the manifold is even-dimensional and the connection preserves the $\mathbb{Z}_2$-grading then the Dirac operator is of the form* (3.69).

3.13. Boundary behaviour.

It is of fundamental importance to examine the behaviour of the Dirac operator at the boundary.

First note that there is no completely natural way that a general b-connection induces a connection on the boundary. This corresponds to $T\partial X$ being a quotient of ${}^bT_{\partial X}X$ rather than a subbundle. However an exact b-metric (see Exercise 2.9) gives an embedding (as the orthocomplement to $x\partial_x$) $T\partial X \longrightarrow {}^bT_{\partial X}X$ which induces a Lie algebra homomorphism on the spaces of sections, so ${}^bT^*_{\partial X}X \longrightarrow T^*\partial X$ can be taken as orthogonal projection. If $V \in \mathcal{C}^\infty(\partial X; T\partial X)$ let $\widetilde{V} \in \mathcal{C}^\infty(\partial X; {}^bTX)$ be the image of V. Then if ${}^b\nabla$ is a b-connection on some bundle E set

$$\nabla_V e = {}^b\nabla_{\widetilde{V}} e,\ e \in \mathcal{C}^\infty(\partial X; E).$$

This defines the induced connection on ∂X, corresponding to the choice of an exact b-metric.

EXERCISE 3.33. Check that if ${}^b\nabla$ is actually a true connection this induced connection on the boundary is the usual one. More generally show that a different choice of exact b-metric changes the induced connection by a homomorphism (*cf.* (2.97))

$$d\psi \otimes {}^b\nabla_{x\partial/\partial x},\ \psi \in \mathcal{C}^\infty(\partial X).$$

As in (3.41) the action of $\eth_E$ can be written in terms of any local orthonormal frame, v_j and dual coframe ϕ^j :

$$\eth_E u = -i \sum_j \mathrm{cl}(\phi^j)^b\nabla_{v_j} u. \tag{3.70}$$

Recall that over the boundary the spin bundle in the even dimensional case splits as the direct sum of two copies of the induced spin bundle. In fact using the identification (3.30) of the Clifford algebra over the boundary with the positive part of the Clifford algebra of the manifold, restricted to the boundary

$$\mathrm{Cl}(\partial X) \longrightarrow \mathrm{Cl}^+_{\partial X} X,\ \phi^j \longmapsto i\phi^j \cdot \frac{dx}{x}, \tag{3.71}$$

the spin bundle over the boundary, S_0, can be identified with ${}^+S_{\restriction\partial X}$. Denote this identification

$$M_+ : {}^+S_{\restriction\partial X} \longrightarrow S_0.$$

Then the other half of the spin bundle can be identified with S_0 through Clifford multiplication:

$$M_-^{-1} = -\,\mathrm{cl}(i\frac{dx}{x})M_+^{-1} : S_0 \longleftrightarrow {}^-S_{\restriction\partial X}.$$

Thus $M_-^{-1}M_+ = -i\,\mathrm{cl}(\frac{dx}{x})$. As an operator on ${}^+S_{\restriction\partial X}$ one then finds

$$M_+\eth_0 M_+^{-1} = \sum_{j=1}^n i\,\mathrm{cl}(\phi^j)\,\mathrm{cl}(\frac{dx}{x})\nabla_{iv_j},\ N = n+1.$$

LEMMA 3.34. *With these identifications, the restriction to the boundary of the twisted Dirac operator $\eth_E^-$ for an even-dimensional exact b-spin manifold and coefficient bundle with (true) connection is given in terms of the twisted Dirac operator for the induced spin structure and connection on the boundary by*

$$(\eth_E u)_{\restriction\partial X} = \begin{pmatrix} 0 & M_+^{-1}\eth_{0,E}M_- \\ M_-^{-1}\eth_{0,E}M_+ & 0 \end{pmatrix} (u_{\restriction\partial X}) \tag{3.72}$$

for all $u \in \mathcal{C}^\infty(X; S \otimes E)$.

PROOF: As usual the orthonormal basis should be chosen with first element reducing to $x\partial_x$ at the boundary. The assumption that the connection on E is a true connection means that ${}^b\nabla_{x\partial_x}$ vanishes at ∂X and then (3.72) follows directly from (3.70) and (3.71).

In the case of a general b-connection on E formula (3.72) becomes

$$(\eth_E u)_{\restriction\partial X} = \begin{pmatrix} 0 & M_+^{-1}\eth_{0,E}M_- \\ M_-^{-1}\eth_{0,E}M_+ & 0 \end{pmatrix} (u_{\restriction\partial X}) . \tag{3.73}$$

Indeed the same approach works in the case of a graded Hermitian module. Define

$$E_0 = E^+_{\restriction\partial X},$$

with Clifford action given by (3.71). Then the b-connection induces a connection on E_0 and:

COROLLARY. *The same result,* (3.73), *holds for the case of the generalized Dirac operators in Lemma 3.32 provided $\eth_{0,E}$ is the generalized Dirac operator induced on the boundary Clifford module E_0.*

EXERCISE 3.35. Check the details of the proof of this corollary.

Later this result will be extended somewhat, once the indicial homomorphism has been defined, to arrive at the first two parts of (In.23).

3.14. Dirac operators of warped products.

We proceed to compute the form of the Dirac operator on a warped product, as considered in §2.10.

Consider the product case. Thus suppose that M is an odd-dimensional spin manifold, with metric h and spinor bundle S_0, and consider the even-dimensional manifold $X = \mathbb{S}^1 \times M$ with the product metric $d\theta^2 + h$. The discussion in the preceding two sections applies with minor changes. So X has a spin structure the spinor bundle of which can be identified with $S_0 \oplus S_0$, where $M_+ : {}^+S \equiv S_0$ is regarded as an equality and $M_- : {}^-S \longleftrightarrow S_0$ is given by $M_- = -iM_+ \operatorname{cl}(d\theta)$. Clifford multiplication, cl_0, on S_0 and $S = {}^+S \oplus {}^-S$ are related as in (3.71), i.e.

$$\operatorname{cl}_0(\eta) = -i \operatorname{cl}(d\theta) \operatorname{cl}(\eta), \tag{3.74}$$

where in the right $\eta \in T^*M$ is regarded as a 1-form on X. The Dirac operator on X then becomes

$$\eth = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \frac{1}{i}\frac{\partial}{\partial\theta} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \eth_0. \tag{3.75}$$

To generalize this formula to the metric (2.71) we first need to identify the spin bundle over X with the bundle in the product case. To define the spin structure for the metric g, the orthonormal coframe bundle of g can be identified with that in the product case. If the η_i for $i = 1, \dots, 2k-1$ are a local orthonormal frame for h on M then $d\theta, \eta_i$ is a local orthonormal frame for X with the product metric and $d\theta, e^{\phi}\eta_i$ an orthonormal frame for g. This gives a local identification of the orthonormal coframe bundles; since it is independent of the choice of the η_i it extends globally. This identification extends to define a spin structure for g by identifying the spin (structure) bundle of it with the product case. Since the spinor bundles are associated bundles of these principal bundles they too are identified. Thus the Dirac operator for the metric g acts on the same bundle as (3.75), so the two operators can be compared.

To find the form of the Dirac operator we simply compute expressions for Clifford multiplication $\widetilde{\mathrm{cl}}$ and covariant differentiation, $\widetilde{\nabla}$, for the metric g, on the bundle $S = S_0 \oplus S_0$. Since Clifford multiplication is defined algebraically

$$\begin{aligned} \widetilde{\mathrm{cl}}(\eta) = \mathrm{cl}(e^{-\phi}\eta),\ \ \eta \in \mathcal{C}^\infty(M; T^*M), \\ \widetilde{\mathrm{cl}}(d\theta) = \mathrm{cl}(d\theta). \end{aligned} \tag{3.76}$$

Now consider a point $p \in M$ and fix local coordinates z^j near p for which the differentials are orthonormal at p and covariant constant there, all with respect to the metric h. Then combining (3.67) and (2.73)

$$\begin{aligned} \widetilde{\nabla}_{\partial_\theta} s &= \frac{\partial s}{\partial \theta} - \frac{2k-1}{4}\frac{\partial \phi}{\partial \theta} e^{-2\phi} s \\ \widetilde{\nabla}_j s &= \nabla_j s + \frac{1}{4}\frac{\partial \phi}{\partial \theta} e^{\phi}\, \mathrm{cl}(dz^j)\, \mathrm{cl}(d\theta). \end{aligned} \tag{3.77}$$

Inserting these formulæ into the definition of the Dirac operator gives:

Lemma 3.36. *With the identification above of the spinor bundle for the metric (2.71) on $X = \mathbb{S}^1 \times M$ with the spinor bundle in case $\phi \equiv 0$ the Dirac operator becomes*

$$\widetilde{\eth} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \left\{ \frac{1}{i}\frac{\partial}{\partial \theta} + F(\theta) \right\} + e^{-\phi} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \eth_0, \tag{3.78}$$

where $F(\theta)$ is a function of θ.

This formula is used in Chapter 8 in the proof of the absolute convergence of the integral defining the eta invariant.

Exercise 3.37. Show that

$$F(\theta) = \frac{i(2k-1)}{4}\frac{\partial \phi}{\partial \theta}(e^{-2\phi} - 1). \tag{3.79}$$

Chapter 4. Small b-calculus

The next major topic is the calculus of b-pseudodifferential operators, since this allows the analytic properties of elliptic b-differential operators to be examined. The treatment starts very geometrically. In particular Figure 2 of Chapter 1 will be explained. Recall from (1.24) and (1.26) that the kernels of the inverses of operators such as ${}^b\eth_+^* {}^b\eth_+ + 1$ can be expected to be simplest when expressed in terms of the singular coordinates (1.25). In the one-dimensional case this is rather a minor issue. In the higher-dimensional examples of interest here it is more significant, as there will be a countably infinite superposition of terms like (1.26), with different constants c. To handle these systematically the properties of the coordinate change (1.25) will be examined with some care.

4.1. Inward-pointing spherical normal bundle.

Let X be a compact manifold with boundary. The first goal is to define X_b^2, obtained from X^2 by the introduction of singular coordinates, so that

$$\frac{x - x'}{x + x'} \in \mathcal{C}^\infty(X_b^2),$$

with x and x' are defining functions for the boundary of X lifted from the respective factors, x from the left and x' from the right. The space will come equipped with a surjective smooth *blow-down* map

$$\beta_b : X_b^2 \longrightarrow X^2. \tag{4.1}$$

To construct X_b^2 as a set (i.e. independent of the choice of coordinates) assume that ∂X is connected and consider

$$B = \partial X \times \partial X = \{x = x' = 0\} \subset X^2. \tag{4.2}$$

This is the manifold which should be replaced by a bigger set in which some information concerning the direction of approach to B is included. Directions are associated to curves, so consider all $\mathcal{C}^\infty$ curves with only their endpoint on B and which are not tangent to B at that point:

$$\chi : [0,1] \longrightarrow X^2, \ \chi(t) \in B \Longleftrightarrow t = 0, \ \chi'(0) \notin T_{\chi(0)}B. \tag{4.3}$$

This space, although not linear, has a natural topology as a subspace of the space of all $\mathcal{C}^\infty$ curves in X^2, corresponding to uniform convergence of all derivatives on compact subsets of coordinate patches.

The equivalence relation placed on these curves is:

$$\chi_1 \sim \chi_2 \Longleftrightarrow \chi_1(0) = \chi_2(0) = p,\ \chi_1'(0) \equiv a\chi_2'(0) \pmod{T_pB},\ a \in \mathbb{R}^+.$$

This identifies two curves if they correspond to the same end point, in B, and have derivatives at that point which are multiples, up to a term tangent to B. Thus the set of equivalence classes is the inward-pointing unit (or really spherical) normal bundle to B :

$$S_+N_p(B) = \{\chi \text{ in } (4.3), \chi(0) = p\}/\sim \tag{4.4}$$

and then

$$S_+N(B) = \bigsqcup_{p\in B} S_+N_p(B)$$

gives a bundle over B with fibre diffeomorphic to $[-1,1]$. This will be proven directly by constructing coordinates on $S_+N(B)$.

LEMMA 4.1. *If $x \in \mathcal{C}^\infty(X)$ is a defining function for ∂X and $\tilde{x} = \pi_L^* x$, $\tilde{x}' = \pi_R^* x$ denote its lifts to X^2 from the left and right factors then for any curve (4.3) the limit*

$$\tau([\chi]) = \lim_{t\downarrow 0} \chi^*\left(\frac{\tilde{x}-\tilde{x}'}{\tilde{x}+\tilde{x}'}\right) \tag{4.5}$$

exists and depends only on $[\chi] \in S_+N(B)$. This function gives an identification

$$S_+N(B) \ni [\chi] \longmapsto (p = \chi(0), \tau([\chi])) \in B \times [-1,1]. \tag{4.6}$$

Another choice of x determines, through (4.6), a diffeomorphism of the form

$$B \times [-1,1] \ni (p,\tau) \longmapsto \left(p, \tau' = \frac{A(p)+\tau}{1+A(p)\tau}\right) \in B \times [-1,1], \tag{4.7}$$

where $A \in \mathcal{C}^\infty(B)$ is given by

$$A(p) = \frac{a(y)-a(y')}{a(y)+a(y')},\ p = (y,y') \in \partial X \times \partial X,\ 0 < a \in \mathcal{C}^\infty(\partial X). \tag{4.8}$$

PROOF: If $\tilde{x}$ and $\tilde{x}'$ are the elements in $\mathcal{C}^\infty(X^2)$ indicated then at each point $p \in B$, $d\tilde{x}$ and $d\tilde{x}'$ are linear coordinates in the two dimensional space $N_pB = T_pX^2/T_pB$. By definition, in (4.4),

$$S_+N_p(B) = \{0 \neq v \in N_pB;\ d\tilde{x}(v) \geq 0, d\tilde{x}'(v) \geq 0\}/\mathbb{R}^+.$$

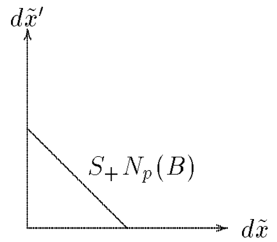

Figure 4. The spherical normal fibre.

Thus there is an identification

$$S_+N_p(B) \simeq \{v \in N_pB;\ d\tilde{x}(v) \geq 0,\ d\tilde{x}'(v) \geq 0, d\tilde{x}(v) + d\tilde{x}'(v) = 1\}.$$

With this identification the function (4.5) is just $\tau(v) = d\tilde{x}(v) - d\tilde{x}'(v) \in [-1,1]$. Under change of the defining function from x to $\bar{x} = a(y)x + O(x^2)$, $0 < a \in \mathcal{C}^\infty(\partial X)$, the new function (4.5) is

$$\tau' = \lim_{t\downarrow 0} \chi^* \left(\frac{\tilde{\bar{x}} - \tilde{\bar{x}}'}{\tilde{\bar{x}} + \tilde{\bar{x}}} \right) = \frac{a(y)(1+\tau) - a(y')(1-\tau)}{a(y)(1+\tau) + a(y')(1-\tau)}.$$

This gives (4.7) and (4.8).

Thus not only does $S_+N(B)$ have a $\mathcal{C}^\infty$ structure, given by (4.6), but the fibres have a *projective* structure. Indeed if $\alpha \in [-1,1]$ then

$$M_\alpha \colon S_+N(B) \longrightarrow S_+N(B),\ M_\alpha\tau = \frac{\alpha+\tau}{1+\alpha\tau} \tag{4.9}$$

is independent of the choice of projective coordinate τ, since

$$\tau' = \frac{A+\tau}{1+A\tau'} \Longrightarrow \frac{\alpha+\tau'}{1+\alpha\tau'} = \frac{A+M_\alpha\tau}{1+AM_\alpha\tau}.$$

Using the coordinate

$$s = \frac{1+\tau}{1-\tau} \in (0,\infty) \text{ on } S_+N(B)\backslash\partial(S_+N(B)) \tag{4.10}$$

the action (4.9) becomes

$$s \longmapsto \beta s, \quad \beta = \frac{1-\alpha}{1+\alpha} \in (0,\infty).$$

That is, the interior of each fibre of $S_+N(B)$ has a natural $(0,\infty)$-action.

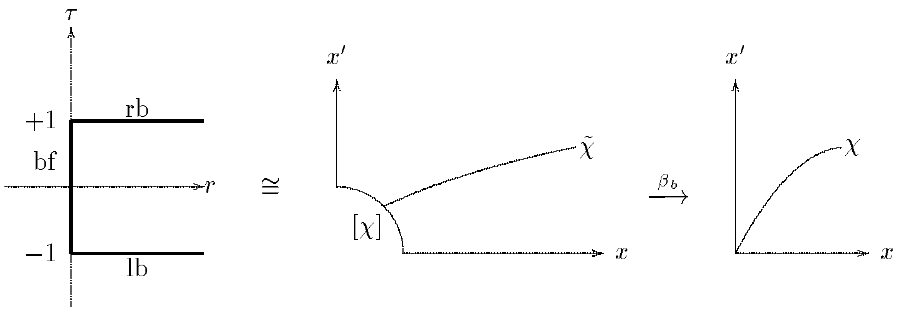

Figure 5. $\beta_b \colon X_b^2 \longrightarrow X^2$.

4.2. The b-stretched product.

If ∂X is connected the b-stretched product is obtained by replacing B by $S_+N(B)$ in X^2, i.e.

$$X_b^2 = S_+N(B) \sqcup (X^2 \backslash B).$$

This is certainly well-defined as a set and (4.1) arises by letting β_b act as the identity on $X^2\backslash B$ and taking it to be the projection from $S_+N(B)$ to B, $\beta_b([\chi]) = \chi(0) \in B$. It remains to show that X_b^2 has a natural $\mathcal{C}^\infty$ structure with respect to which β_b is $\mathcal{C}^\infty$. Fix the topology on X_b^2 by noting that each curve χ in (4.3) defines a map

$$\tilde{\chi} \colon [0,1] \longrightarrow X_b^2,\ \tilde{\chi}(t) = \chi(t),\ t > 0,\ \tilde{\chi}(0) = [\chi].$$

A subset $U \subset X_b^2$ is open if it meets $X^2 \setminus B$ and $S_+N(B)$ in open sets and provided that whenever χ is a curve, as in (4.3), such that $\tilde{\chi}(0) \in U$ there exists $\epsilon > 0$ and a neighbourhood of χ amongst such curves for which $\tilde{\mu}(t) \in U$ for all $0 \le t < \epsilon$ and for all curves μ in the neighbourhood. It would actually be enough to take the $\mathcal{C}^1$ topology on curves to get the same topology on X_b^2. Notice in particular that the function $\tau = (\tilde{x} - \tilde{x}')/(\tilde{x} + \tilde{x}')$ on $X^2\backslash B$, extended to $S_+N(B)$ by (4.5), is in $\mathcal{C}^0(X_b^2)$. Then define $\mathcal{C}^\infty(X_b^2) \subset \mathcal{C}^0(X_b^2)$ as consisting of those functions which can be expressed locally near each point of $S_+N(B)$ in the form $F(\tau, \tilde{x}, \tilde{y}, \tilde{x}', \tilde{y}')$, with F a $\mathcal{C}^\infty$ function, i.e. $\mathcal{C}^\infty(X_b^2)$ is generated locally by τ and $\beta_b^*\mathcal{C}^\infty(X^2)$. As in the discussion above, under a change of defining function to $\bar{x} = a(x,y)x$, $0 < a \in \mathcal{C}^\infty(X)$, τ is transformed to

$$\bar{\tau} = \frac{A(\tilde{x}, \tilde{y}, \tilde{x}', \tilde{y}') + \tau}{1 + A(\tilde{x}, \tilde{y}, \tilde{x}', \tilde{y}')\tau}$$

where
$$A(\tilde{x},\tilde{y},\tilde{x}',\tilde{y}') = \frac{a(\tilde{x},\tilde{y}) - a(\tilde{x}',\tilde{y}')}{a(\tilde{x},\tilde{y}) + a(\tilde{x}',\tilde{y}')}.$$

Thus $\bar{\tau} \in \mathcal{C}^\infty(X_b^2)$ and the definition is indeed independent of the choice of x. As illustrated in Figure 5 there is a neighbourhood of $S_+N(B)$ in X_b^2 of the form $[-1,1] \times [0,\epsilon) \times \partial X \times \partial X$. In fact for any boundary defining function $x \in \mathcal{C}^\infty(X)$ the set $G = \{\tilde{x} + \tilde{x}' < \epsilon\}$ is, for small $\epsilon > 0$, such a neighbourhood. The functions τ and $\tilde{x} + \tilde{x}'$ on G give the desired decomposition. It follows that X_b^2 is a compact manifold with corners (just as is X^2) with the blow-down map, β_b in (4.1), $\mathcal{C}^\infty$. The construction of X_b^2 from X^2 by blowing up B in (4.2) is a special case of the general process of (real) blow-up of a submanifold. In the notation of [63] or [31]
$$X_b^2 = [X^2; B]$$
but the general construction is not needed here.

So far the construction has been carried out under the assumption that ∂X consists of one component manifold. If the boundary has several components:

$$\partial X = Y_1 \sqcup Y_2 \sqcup \cdots \sqcup Y_r \tag{4.11}$$

then set

$$B = \bigsqcup_{j=1}^{r} (Y_j \times Y_j) \subset \partial X \times \partial X. \tag{4.12}$$

The b-stretched product X_b^2 is defined by blowing up B, i.e. by blowing up each of the components of B in (4.12).

EXERCISE 4.2. Make sure that you understand that 'blow-up' is just the introduction of polar coordinates around the submanifold. In particular check that $r = (\tilde{x}^2 + (\tilde{x}')^2)^{\frac{1}{2}}$ and $\theta = \arctan(\tilde{x}/\tilde{x}')$ are $\mathcal{C}^\infty$ functions on X_b^2. How would you express the naturality of the $\mathcal{C}^\infty$ structure on X_b^2 from this point of view, i.e. what would you have to do to prove it?

4.3. Submanifolds of X_b^2.

It is important to gain a clear understanding of the geometry of X_b^2, which will be the carrier for the Schwartz kernels of the operators in the b-calculus. The boundary hypersurfaces of X_b^2 will be denoted lb, rb and bf, where

$$\mathrm{bf}(X_b^2) = S_+N(B) \tag{4.13}$$

is the 'new' or 'front face' replacing B. The left and right boundary faces, lb and rb, are just the 'old' boundary faces:

$$\mathrm{lb}(X_b^2) = \mathrm{cl}\big(\beta_b^{-1}(\partial X \times \overset{\circ}{X})\big),\ \mathrm{rb}(X_b^2) = \mathrm{cl}\big(\beta_b^{-1}(\overset{\circ}{X} \times \partial X)\big) \text{ in } X_b^2. \tag{4.14}$$

Notice that $\partial X \times \overset{\circ}{X} = (\partial X \times X) \setminus B$. The lift is defined in this way, as the closure in X_b^2 of the part of the submanifold (in this case $\mathrm{lb}(X^2) = \partial X \times X$) outside B, rather than just the preimage because

$$\beta_b^{-1}(\mathrm{lb}(X^2)) = \mathrm{bf}(X_b^2) \cup \mathrm{lb}(X_b^2)$$

and the front face is not to be thought of as part of the left boundary. Indeed the main idea is to separate the left and right boundaries. Observe that (provided ∂X is connected)

$$\mathrm{lb}(X_b^2) \cap \mathrm{rb}(X_b^2) = \emptyset.$$

There is another submanifold of primary importance in the analysis of differential operators, namely the diagonal

$$\Delta = \{(p,p) \in X^2;\ p \in X\} \simeq X.$$

The *lifted* or b-diagonal is defined, as in (4.14), by setting

$$\Delta_b = \mathrm{cl}(\beta_b^{-1}(\Delta \backslash B)) \text{ in } X_b^2. \tag{4.15}$$

LEMMA 4.3. *The lifted diagonal is an embedded, closed submanifold of X_b^2 which is diffeomorphic to X under β_b and which meets the boundary of X_b^2 only in* $\mathrm{bf}(X_b^2)$, *which it intersects transversally.*

PROOF: Of course in $X^2 \backslash B$ nothing has been changed, so it is only necessary to consider a neighbourhood of $\mathrm{bf}(X_b^2)$. From the definition of $\tau = (\tilde{x} - \tilde{x}')/(\tilde{x} + \tilde{x}')$ in $X^2 \backslash B$ it follows, by continuity, that $\tau \equiv 0$ on Δ_b. Similarly $\tilde{y} = \tilde{y}'$ on Δ_b for any local coordinates in X. This gives $\dim X$ independent defining functions and shows that Δ_b is an embedded submanifold. The local coordinates x, y lift to local coordinates $\tilde{x}, \tilde{y}$ on Δ_b, so $\beta_b \colon \Delta_b \longleftrightarrow X$ is a diffeomorphism. Notice that

$$\mathrm{bf}(X_b^2) = \{r = 0\},\ r = \tilde{x} + \tilde{x}' \in \mathcal{C}^\infty(X_b^2).$$

Since $dr = 2d\tilde{x}$ on Δ_b is non-vanishing,

$$\begin{gathered}\mathrm{bf}(X_b^2) \pitchfork \Delta_b = \\ \{[\chi] \in S_+N(B); \tau([\chi]) = 0,\ p = \chi(0) \in \Delta_{\partial X} \subset \partial X \times \partial X\}.\end{gathered}$$

This also completes the explanation of Figure 2 in Chapter 2.

4.4. Lifting vector fields.

What has been accomplished by the introduction of X_b^2? Later it will be seen to lead to a simplification of the Schwartz kernels of b-differential operators and this simple characterization makes the definition of the b-pseudodifferential operators reasonably obvious. The main step in doing this is to see what happens to the elements of $\mathcal{V}_b(X)$ when they are lifted to X^2, to act just on the left factor, and then are further lifted to X_b^2. In general one cannot lift a vector field under a $\mathcal{C}^\infty$ map, however:

Proposition 4.4. *Each element of $\mathcal{V}_b(X)$, as a $\mathcal{C}^\infty$ vector field on the left factor of X in X^2, lifts to a $\mathcal{C}^\infty$ vector field on X_b^2.*

Proof: As already observed, near each point of $\mathrm{bf}(X_b^2)$ the coordinates

$$\tau = \frac{x - x'}{x + x'}, r = x + x', y, y' \tag{4.16}$$

can be used, where (x, y) are coordinates in the left factor and (x', y') are coordinates in the right factor, with $x = x' \in \mathcal{C}^\infty(X)$. From now on the rather heavy-handed tilde notation used above to distinguish between a function on X and the lift to one of the factors of X^2 will be dropped. A general element of $\mathcal{V}_b(X)$ is of the form

$$V = \alpha_0 x \frac{\partial}{\partial x} + \sum_{j \geq 1} \alpha_j \frac{\partial}{\partial y_j},$$

with the coefficients, α_l, $\mathcal{C}^\infty$ functions in local coordinates. Since the coefficients certainly lift to be $\mathcal{C}^\infty$ and the $\partial/\partial y_j$ lift to $\partial/\partial y_j$ only the lift of $x\partial/\partial x$ needs to be examined. From (4.16),

$$x\frac{\partial}{\partial x} = x\frac{\partial}{\partial r} + \frac{2xx'}{(x+x')^2}\frac{\partial}{\partial \tau} = \frac{1}{2}(1+\tau)r\frac{\partial}{\partial r} + \frac{1}{2}(1-\tau^2)\frac{\partial}{\partial \tau} \tag{4.17}$$

which is $\mathcal{C}^\infty$, proving the proposition.

Notice what has happened in (4.17). The vector field $x\partial/\partial x$ vanishes at $\mathrm{lb}(X^2)$, i.e. $\{x = 0\}$. When lifted to X_b^2, near $\mathrm{bf}(X_b^2)$, it again vanishes at $\mathrm{lb}(X_b^2)$, but not on $\mathrm{bf} \setminus \mathrm{lb}$. Indeed the lifts of the $\partial/\partial y_j$ and $x\partial/\partial x$ are independent at each point of Δ_b and no non-trivial linear combination of them is tangent to Δ_b. This can be reexpressed in the form:

Lemma 4.5. *The lift to X_b^2 of $\mathcal{V}_b(X)$ from the left factor of X^2 is a Lie subalgebra of $\mathcal{V}_b(X_b^2)$ which is transversal to Δ_b.*

At each point of Δ_b the normal space is

$$N_p(\Delta_b) = T_pX_b^2/T_p\Delta_b,\ p \in \Delta_b. \tag{4.18}$$

Thus the values at p of the elements of $\mathcal{V}_b(X)$ define a subspace of $T_pX_b^2$ of dimension equal to that of X. By Lemma 4.5 this space is transversal to Δ_b and can be identified with ${}^bT_qX$, where $\beta_b(p) = (q,q)$. Thus

$$T_pX_b^2 = T_p\Delta_b \oplus {}^bT_qX,\ \beta_b(p) = (q,q),\ p \in \Delta_b. \tag{4.19}$$

Combined with (4.18) this gives:

LEMMA 4.6. *The normal bundle to Δ_b in X_b^2 is naturally isomorphic to ${}^bTX$ and dually*

$$N^*\Delta_b \cong {}^bT^*X. \tag{4.20}$$

This turns out to be important in the discussion of the symbol mapping.

EXERCISE 4.7. Prove (4.20) directly, in the spirit of b-geometry (see [63] for more details). First show that the composite map $\pi_{b,L}^2 = \pi_L^2 \cdot \beta_b : X_b^2 \longrightarrow X$, where $\pi_L^2 : X^2 \longrightarrow X$ is projection onto the left factor, is a b-map, meaning that if ρ is a defining function for the one boundary hypersurface of X then

$$(\pi_{b,L}^2)^*\rho = \rho_{\mathrm{bf}}^{e(\mathrm{bf})}\rho_{\mathrm{rb}}^{e(\mathrm{rb})}\rho_{\mathrm{lb}}^{e(\mathrm{lb})}, \tag{4.21}$$

where the ρ_F's are defining functions for the various boundary hypersurfaces of X_b^2. In this case $e(\mathrm{bf}) = 1$, $e(\mathrm{rb}) = 0$ and $e(\mathrm{lb}) = 1$. The reason for considering such a condition as (4.21) is that it implies that under pull-back smooth b-forms become smooth b-forms. In this case

$$(\pi_{b,L}^2)^*\frac{d\rho}{\rho} = \frac{d\rho_{\mathrm{bf}}}{\rho_{\mathrm{bf}}} + \frac{d\rho_{\mathrm{lb}}}{\rho_{\mathrm{lb}}}. \tag{4.22}$$

Use this to see (4.20).

Having gone this far it is probably worthwhile to introduce the most important notion described in [63] for maps between manifolds with corners, namely the notion of a b-fibration. This will not be used below (but makes for a much cleaner proof of the composition formula, etc.) For b-maps the differential extends, by continuity from the interior of the manifold, to a b-differential

$$(\pi_{b,L}^2)_* : {}^bT_pX_b^2 \longrightarrow {}^bT_qX,\ q = \pi_{b,L}^2(p). \tag{4.23}$$

Indeed this is just the dual statement to the pull-back on ${}^bT^*X^2$ which is expressed by (4.22). There are two properties of (4.23) which are not possessed by general b-maps. First

(4.24) The b-differential is surjective,

which is easy enough to check. A map with this property is called a b-submersion. So check that $\pi^2_{b,L}$ is a b-submersion. In addition there is another, independent, condition satisfied by $\pi^2_{b,L}$. Recall that at each point of $q \in X^2_b$ (or any manifold with corners) the b-normal space to the boundary is a well-defined subspace of ${}^bT_qX^2_b$, just given by the values at that point of the vector fields which vanish at the boundary in the ordinary sense (i.e. is spanned by $x\partial/\partial x$ at a boundary point of X.) It is easy to see that the b-differential of a b-map must always map the b-normal space at any point into the b-normal space at the image. The map is said to be b-normal if

(4.25) The b-differential is surjective as a map between b-normal spaces.

Check that $\pi^2_{b,L}$ has this property too. A b-map with the two properties (4.24) and (4.25), i.e. a b-submersion which is b-normal, is called a b-fibration in [63]. Thus $\pi^2_{b,L}$ is a b-fibration. Why should you care? Part of the answer to this can be found in Chapter 5 (or [62], better, in [63] of course). Roughly speaking a b-fibration has properties analogous to that of a fibration, especially as regards the push-forward of distributions.

Check that the blow-down map β_b is a b-submersion which is not b-normal, hence not a b-fibration. Give an example of a b-map which is not a b-submersion but is b-normal.

So far only the lift of $\mathcal{V}_b(X)$ from the left factor has been examined. To avoid accusations of 'handism' (oddhandedness??) simply consider the natural involution on X^2 :

$$I\colon X^2 \ni (p,p') \longmapsto (p',p) \in X^2.$$

This fixes each point of B and (therefore) lifts to a diffeomorphism, I_b, of X^2_b. Indeed, in the local coordinates (τ, r, y, y') near $\mathrm{bf}(X^2_b)$

$$I_b\colon X^2_b \longrightarrow X^2_b,\ I_b^2 = \mathrm{Id},\ I_b^*\tau = -\tau, I_b^*r = r, I_b^*y = y'.$$

Of course I_b interchanges lb and rb and shows that the discussion of the lift from the left extends immediately to the lift from the right.

EXERCISE 4.8. Show that the same isomorphism, (4.20), arises by using the decomposition analogous to (4.19) with $\mathcal{V}_b(X)$ the lift from the right factor.

EXERCISE 4.9. Justify the 'therefore' in parenthesis above by checking that under the blow-down map of a submanifold (or just for β_b) any $\mathcal{C}^\infty$ vector field tangent to the submanifold blown-up (in this case B) lifts to be $\mathcal{C}^\infty$ on the blown-up manifold (see also [63]).

The simultaneous lift from left and right is also something to consider. If V and W are vector bundles over X let $V \boxtimes W$ be the exterior tensor product, namely the bundle over X^2 with fibres $V_p \otimes W_q$ over $(p,q) \in X^2$. Then it is clear that

$$^bTX \boxtimes {}^bTX \cong {}^bT(X^2)$$

is a vector bundle on X^2 with the property analogous to that for which ^bTX was defined in the first place:

$$\begin{aligned}\mathcal{V}_b(X^2) &= \{V \in \mathcal{C}^\infty(X^2; TX); V \text{ is tangent to } \mathrm{rb} \cup \mathrm{lb}\} \\ &\cong \mathcal{C}^\infty(X^2; {}^bTX^2).\end{aligned}$$

Another way of seeing this is to note that the sum of the lifts from left and right of $\mathcal{V}_b(X)$ spans $\mathcal{V}_b(X^2)$ over $\mathcal{C}^\infty(X^2_b)$.

LEMMA 4.10. *The lift to X^2_b of $\mathcal{V}_b(X^2)$ spans*

$$\mathcal{V}_b(X^2_b) = \{V \in \mathcal{C}^\infty(X^2_b; TX^2_b); V \textit{ is tangent to } \mathrm{lb} \cup \mathrm{rb} \cup \mathrm{bf}\}$$

over $\mathcal{C}^\infty(X^2_b)$ and therefore

$$\mathcal{V}_b(X^2_b) = \mathcal{C}^\infty(X^2_b; {}^bTX^2_b) \textit{ with } {}^bTX^2_b \cong \beta_b^*({}^bTX^2). \tag{4.26}$$

PROOF: Away from $\mathrm{bf}(X^2_b)$ there is nothing to show because β_b is a diffeomorphism there. Consider local coordinates x, y in the left factor and x', y' in the right factor of X and corresponding coordinates (4.16) near the front face of X^2_b. Then the tangential vector fields $\partial/\partial y_j$ and $\partial/\partial y'_j$ lift to the same vector fields; the lift of $x\partial/\partial x$ is given by (4.17) and similarly

$$x'\frac{\partial}{\partial x'} = x'\frac{\partial}{\partial r} - \frac{2xx'}{(x+x')^2}\frac{\partial}{\partial \tau} = \frac{1}{2}(1-\tau)r\frac{\partial}{\partial r} - \frac{1}{2}(1-\tau^2)\frac{\partial}{\partial \tau}. \tag{4.27}$$

The sum of (4.27) and (4.17) is $r\partial/\partial r$ so the span of the lifted vector fields is the same as that of

$$r\frac{\partial}{\partial r},\ (1-\tau^2)\frac{\partial}{\partial \tau},\ \frac{\partial}{\partial y_j} \text{ and } \frac{\partial}{\partial y'_j}.$$

Since these vector fields clearly span $\mathcal{V}_b(X^2_b)$, locally, this proves the lemma.

EXERCISE 4.11. Extend Lemma 4.5 to show that elements of $\mathrm{Diff}_b^m(X^2)$ lift from X^2 to X_b^2 under β_b, defining a map

$$\beta_b^*\colon \mathrm{Diff}_b^m(X^2) \longrightarrow \mathrm{Diff}_b^m(X_b^2). \tag{4.28}$$

Show further that the range of this map spans $\mathrm{Diff}_b^m(X_b^2)$ over $\mathcal{C}^\infty(X_b^2)$.

4.5. Densities.

The b-stretched product just discussed is the starting point for the description of the b-calculus on a compact manifold with boundary, i.e. the calculus of b-pseudodifferential operators. This calculus is designed to include the generalized inverses of elliptic b-differential operators and not too much more. The term *calculus* is used to indicate that the b-pseudodifferential operators do *not* form an algebra. However, when composition of two such operators is possible the composite is in the calculus. The first step is to define the *small* calculus of b-pseudodifferential operators which *is* an algebra.

First a few words about densities, and in particular b-densities. As a general principle, from now on almost all analytic discussion will be carried out for operators acting on b-half-densities, in order to reduce the book-keeping overhead associated with bundles. The extension to the general case is then mainly a matter of notation. Recall that if V is a vector space of dimension n then the space of s-densities on V is, for $s \in \mathbb{R}$,

$$\Omega^s V = \{u\colon \Lambda^n V^* \setminus \{0\} \longrightarrow \mathbb{R}; u(t\alpha) = |t|^s u(\alpha) \quad \forall\, \alpha \in \Lambda^n V^*, t \neq 0\}.$$

Thus $\mu \in \Omega^s V$ is fixed by its value at any one $0 \neq \alpha \in \Lambda^n V^*$ and so $\Omega^s V$ is always a one-dimensional vector space with an orientation, fixed by $u(\alpha) > 0$. Directly from the definition there are canonical isomorphisms:

$$\begin{aligned} \Omega^s V \otimes \Omega^t V &\equiv \Omega^{s+t} V \quad \forall\ s,t \in \mathbb{R} \\ \Omega^0 V &\equiv \mathbb{R} \\ \Longrightarrow \Omega^{-s} V &\equiv (\Omega^s V)^*. \end{aligned} \tag{4.29}$$

Similarly if V and W are any finite dimensional vector spaces there is a canonical isomorphism

$$\Omega^s(V \oplus W) \equiv \Omega^s(V) \otimes \Omega^s(W)\ \forall\ s \in \mathbb{R}; \tag{4.30}$$

this just arises from $\Lambda^{n+m}(V^* \oplus W^*) \equiv \Lambda^n V^* \otimes \Lambda^m W^*$ if $\dim V = n$ and $\dim W = m$.

If X is a compact manifold (with or without boundary) then the spaces

$$\Omega_x^s = \Omega^s(T_x^* X)$$

form $\mathcal{C}^\infty$ bundles over X. Note that, being oriented line bundles, the density bundles are always trivial. Only for $s = 0$ is Ω^s canonically trivial.

EXERCISE 4.12. Show that

$$\Omega^s X = \bigsqcup_{x \in X} \Omega^s_x$$

is a bundle associated to the coframe bundle of X.

Most importantly 1-densities, usually just called densities with Ω^1 denoted simply as Ω, can be integrated. The *integral* is then a well-defined linear functional

$$\int_X : \mathcal{C}^\infty(X;\Omega) \longrightarrow \mathbb{R}. \tag{4.31}$$

EXERCISE 4.13. Local coordinates in X induce a basis of $T^*_x X$, hence the basis element $\partial_{x_1} \wedge \cdots \wedge \partial_{x_n}$ in $\Lambda^n(T_x X)$. Show that

$$|dx|^s(a\partial_{x_1} \wedge \cdots \wedge \partial_{x_n}) = |a|^s, \ \ s \in \mathbb{R}$$

is a local basis element of $\Omega^s X$. Use this to show that (4.31) can be defined as the Riemann integral in local coordinates. Thus if $f \in \mathcal{C}^\infty(X;\Omega)$ has support in a coordinate patch, so $f = g(x)|dx|$ in terms of the coordinate trivialization with g a $\mathcal{C}^\infty$ function, then

$$\int_X f = \int_{\mathbb{R}^n} g(x)dx. \tag{4.32}$$

Check that (4.32) is independent of the coordinate system containing the support of f and hence that there is a unique functional (4.31) extending (4.32) by linearity.

The most important fractional densities here are the half-densities. From (4.29) there is a product

$$\mathcal{C}^\infty(X;\Omega^{\frac{1}{2}}) \cdot \mathcal{C}^\infty(X;\Omega^{\frac{1}{2}}) \longrightarrow \mathcal{C}^\infty(X;\Omega). \tag{4.33}$$

This extends to complex sections (i.e. sections of the complexified bundles) and hence gives a sesquilinear pairing:

$$\langle u, v\rangle = \int_X u\overline{v}, \ \ u, v \in \mathcal{C}^\infty(X;\Omega^{\frac{1}{2}}). \tag{4.34}$$

If $\mathcal{C}^\infty(X;\Omega^{\frac{1}{2}})$ is completed with respect to this pairing the result is the Hilbert space of square-integrable half-densities, $L^2(X;\Omega^{\frac{1}{2}})$. For example the Hilbert-Schmidt operators (which will be useful later) can be easily described. Thus:

$$K \in L^2(X^2; \Omega^{\frac{1}{2}}) \Longrightarrow A_K \colon L^2(X; \Omega^{\frac{1}{2}}) \longrightarrow L^2(X; \Omega^{\frac{1}{2}}), \tag{4.35}$$
$$A_K u(x) = \int\limits_X K(x,y)u(y).$$

Here if $\phi \in L^2(X; \Omega^{\frac{1}{2}})$

$$\langle A_K u, \phi\rangle = \int\limits_{X^2} K(x,y)\phi(x)u(y) = \langle K, \phi \boxtimes u\rangle$$

using the (fairly obvious) form of Fubini's theorem

$$L^2(X; \Omega^{\frac{1}{2}}) \boxtimes L^2(X; \Omega^{\frac{1}{2}}) \subset L^2(X^2; \Omega^{\frac{1}{2}}).$$

This can be generalized to operators on distributions. The pairing (4.33) invites the definition of the space of *distributional* half-densities as consisting of the continuous linear maps

$$u \colon \mathcal{C}^\infty(X; \Omega^{\frac{1}{2}}) \longrightarrow \mathbb{C} \text{ continuous and linear } \Longleftrightarrow u \in \mathcal{C}^{-\infty}(X; \Omega^{\frac{1}{2}}). \tag{4.36}$$

Here continuity is with respect to the topology of uniform convergence of all derivatives of the half-density on compact subsets of coordinate patches. Then the pairing (4.33) gives an inclusion

$$\mathcal{C}^\infty(X; \Omega^{\frac{1}{2}}) \hookrightarrow \mathcal{C}^{-\infty}(X; \Omega^{\frac{1}{2}})$$

and in fact the range is dense in the weak topology. For a general vector bundle, E, over X define

$$\mathcal{C}^{-\infty}(X; E) = (\mathcal{C}^\infty(X; E' \otimes \Omega))',$$

where E' is the dual bundle. This is consistent with (4.36) since $(\Omega^{\frac{1}{2}})' \equiv \Omega^{-\frac{1}{2}}$ from (4.29) so $(\Omega^{\frac{1}{2}})' \otimes \Omega \equiv \Omega^{\frac{1}{2}}$.

The general case of a continuous linear operator

$$A \colon \mathcal{C}^\infty(X; E) \longrightarrow \mathcal{C}^{-\infty}(Y; F), \tag{4.37}$$

where E is a $\mathcal{C}^\infty$ vector bundle over X and F is a $\mathcal{C}^\infty$ vector bundle over Y, is covered by Schwartz' kernel theorem:

THEOREM 4.14. *(Schwartz) The continuous linear maps* (4.37) *are in* 1-1 *correspondence with the distributions*

$$K \in \mathcal{C}^{-\infty}\left(Y \times X; F \boxtimes [E' \otimes \Omega(X)]\right),$$

where $A \longleftrightarrow K$ *if*

$$\langle Au, \phi\rangle = \langle K, \phi \boxtimes u\rangle \quad \forall\ u \in \mathcal{C}^\infty(X; E),\ \ \phi \in \mathcal{C}^\infty(Y; F').$$

Quite straightforward proofs of this result are now available (using the Fourier transform) see [46].

For $E = F = \Omega^{\frac{1}{2}}$ this simplifies as in (4.35), i.e.

$$\begin{aligned}\left\{A\colon \mathcal{C}^\infty(X;\Omega^{\frac12}) \longrightarrow \mathcal{C}^{-\infty}(Y;\Omega^{\frac12});\ \text{continuous and linear}\right\}\\ \longleftrightarrow \left\{K \in \mathcal{C}^{-\infty}(Y \times X; \Omega^{\frac12})\right\}.\end{aligned} \tag{4.38}$$

The fact that only half-density bundles appear throughout here is one reason for working with them.

4.6. The space of pseudodifferential operators.

Now recall, for later generalization, the definition of the space of pseudodifferential operators on X, $\partial X = \emptyset$. These spaces were defined by Hörmander in [44] and by Kohn and Nirenberg in [49] although closely related singular integral operators had been used before. These operators can be defined in terms of their Schwartz kernels, using (4.38):

$$\Psi^m(X;\Omega^{\frac12}) \longleftrightarrow I^m(X^2, \Delta; \Omega^{\frac12}). \tag{4.39}$$

On the right is the space of conormal distributional sections of $\Omega^{\frac12}$ of order m associated to $\Delta \subset X^2$. In fact $I^m(Z, Y; E) \subset \mathcal{C}^{-\infty}(Z; E)$ is defined for any closed embedded submanifold $Y \subset Z$, any $m \in \mathbb{R}$ and any vector bundle E over Z (see [45]).

DEFINITION 4.15. The space $I^m(Z, Y; E) \subset \mathcal{C}^{-\infty}(Z; E)$ consists of the distributional sections of E satisfying

$$K_{\restriction Z\setminus Y} \in \mathcal{C}^\infty(Z \setminus Y; E) \tag{4.40}$$

and such that in any local coordinates $x_1, \dots, x_p$ in Z with respect to which $Y = \{x_1 = \cdots = x_q = 0\}$ and on the coordinate patch, O, over which E is trivial,

$$K_{\restriction O} = (2\pi)^{-q} \int\limits_{\mathbb{R}^q} e^{ix'\cdot\xi} a(x'', \xi)\,d\xi, \tag{4.41}$$

where $x' = (x_1, \dots, x_q)$, $x'' = (x_{q+1}, \dots, x_p)$ and $a = (a_1, \dots, a_r)$, taking values in $\mathbb{C}^r$ where $E_{\restriction O} \equiv O \times \mathbb{C}^r$, satisfies

$$\begin{gathered} |D^\alpha_{x''} D^\beta_\xi a_i(x'', \xi)| \le C_{\alpha,\beta,K}(1 + |\xi|)^{m' - |\beta|}, \\ x'' \in K \subset\subset O, \xi \in \mathbb{R}^q, i = 1, \dots, r; \end{gathered} \tag{4.42}$$

here $m' = m - q/4 + n/2$, $n = \dim Z$.

The estimates (4.42) are the *symbol estimates* on the amplitude a, introduced in this form by Hörmander [44]. It is important that the combination (4.41), (4.42) is coordinate invariant, see [47]. Thus, given (4.40), if (4.41) holds in a covering by coordinate patches it holds in any other coordinate patch.

There are more refined classes of operators than those with symbols just satisfying the estimates (4.42). Namely the class of one-step polyhomogeneous operators corresponds to symbols having complete asymptotic expansion with integral step:

$$a_i \sim \sum_{k=0}^{\infty} a_{ik}(x'', \xi)|\xi|^{m'-k}. \tag{4.43}$$

Here the coefficient functions a_{ik} are $\mathcal{C}^\infty$ in $\xi \ne 0$ and homogeneous of degree zero and the meaning of (4.43) is that for each N the difference

$$a_{i,(N)}(x'', \xi) = a_i(x'', \xi) - \sum_{k<N} a_{ik}(x'', \xi)|\xi|^{m'-k}$$

satisfies the estimates (4.42) in $|\xi| \ge 1$ with m' replaced by $m' - N$. Again the existence of such an expansion is a condition independent of coordinates.

If $I^m_{\mathrm{os}}(Z, Y; E) \subset I^m(Z, Y; E)$, the subspace of one-step polyhomogeneous conormal distributions, is used in (4.39) the result is what is often called the 'classical' algebra of pseudodifferential operators on a compact manifold:

$$\Psi^m_{\mathrm{os}}(X; \Omega^{\frac12}) \longleftrightarrow I^m_{\mathrm{os}}(X^2, \Delta; \Omega^{\frac12}).$$

The algebra of pseudodifferential operators can be considered as the microlocalization of the algebra of differential operators. Thus for $k \in \mathbb{N}$

$$\mathrm{Diff}^k(X; \Omega^{\frac12}) \subset \Psi^k_{\mathrm{os}}(X; \Omega^{\frac12}).$$

The kernels of differential operators are the smooth Dirac half-densities supported by the diagonal (as discussed in §4.8). The passage from smooth Dirac sections to conormal sections can be viewed as the process of microlocalization. It is the space $\Psi^*_{\mathrm{os}}(X; \Omega^{\frac12})$ which is to be generalized to compact manifolds with boundary, by this same process of microlocalization.

4.7. Distributions.

To carry out the generalization, first briefly consider distributions on these spaces. For a fuller treatment see [63]. As before the discussion is initially restricted to half-densities, but in this case it is preferable to consider b-half-densities. Thus for any $s \in \mathbb{R}$ the oriented vector spaces ${}^b\Omega^s_x X = \Omega^s({}^bT^*_x X)$ combine to form the $\mathcal{C}^\infty$ bundles

$$ {}^b\Omega^s X = \bigsqcup_{x \in X} {}^b\Omega^s_x X. $$

There is a very simple relationship between these bundles and the bundles of s-densities in the ordinary sense, which are also well-defined over a manifold with boundary. Namely:

$$ \nu \in \mathcal{C}^\infty(X; {}^b\Omega^s) \Longleftrightarrow x^s \nu \in \mathcal{C}^\infty(X; \Omega^s). $$

In particular it follows from this that if

$$ \dot{\mathcal{C}}^\infty(X; E) = \{u \in \mathcal{C}^\infty(X; E); u \text{ vanishes to all orders at } \partial X\} \tag{4.44} $$

for any vector bundle E then

$$ \dot{\mathcal{C}}^\infty(X; {}^b\Omega^s) \equiv \dot{\mathcal{C}}^\infty(X; \Omega^s). \tag{4.45} $$

Using (4.45) the integral (4.31) can be transferred to $\dot{\mathcal{C}}^\infty(X; {}^b\Omega)$. The product map (4.33) also extends to

$$ \mathcal{C}^\infty(X; {}^b\Omega^{\frac{1}{2}}) \times \dot{\mathcal{C}}^\infty(X; {}^b\Omega^{\frac{1}{2}}) \longrightarrow \dot{\mathcal{C}}^\infty(X; {}^b\Omega) $$

since if one of the factors vanishes to infinite order at the boundary so does the product. Then the pairing (4.34) extends to

$$ \mathcal{C}^\infty(X; {}^b\Omega^{\frac{1}{2}}) \times \dot{\mathcal{C}}^\infty(X; \Omega^{\frac{1}{2}}) \ni (\phi, \psi) \longmapsto \int_X \phi\overline{\psi}. \tag{4.46} $$

It turns out to be rather important to note that this bilinear form does not extend directly to the whole of the product of $\mathcal{C}^\infty(X; {}^b\Omega^{\frac{1}{2}})$ with itself, since the integral in (4.31) does not extend directly to all of $\mathcal{C}^\infty(X; {}^b\Omega)$. Certainly

$$ \mathcal{C}^\infty(X; \Omega) = x\mathcal{C}^\infty(X; {}^b\Omega) \xrightarrow{\int_X} \mathbb{R} $$

but in general if $u \in \mathcal{C}^\infty(X; {}^b\Omega)$ the local integral diverges logarithmically near the boundary. This is actually the reason that the trace functional is not defined on the operators in (In.18) and why the b-trace functional is introduced below.

Part of the reason that the b-densities are convenient to deal with is that there is a natural restriction map. Recall that there is a canonically trivial b-normal line subbundle ${}^bN\partial X \subset {}^bT_{\partial X}X$, spanned in local coordinates by $x\partial/\partial x$, and the quotient is naturally

$$ {}^bT_{\partial X}X/{}^bN\partial X \equiv T\partial X. $$

EXERCISE 4.16. Check that (4.30) is also valid when $V \subset U$ is a subspace in the sense that there is a natural isomorphism

$$ \Omega^s(U) \equiv \Omega^s(V) \otimes \Omega^s(U/V). $$

Thus for any $s \in \mathbb{R}$

$$ {}^b\Omega^s_{\partial X}X \equiv \Omega^s\partial X \otimes \Omega^s({}^bN\partial X). $$

Since the b-normal bundle is canonically trivial, by (2.9), so is its density bundle, hence there is a natural restriction map:

$$ \mathcal{C}^\infty(X; {}^b\Omega^s) \xrightarrow{\restriction_{\partial X}} \mathcal{C}^\infty(\partial X; \Omega^s). \tag{4.47} $$

EXERCISE 4.17. Show that in local coordinates $x, y_1, \dots, y_n$ near a boundary point the map (4.47) is just

$$ a(x,y)|\frac{dx}{x}dy|^s \longmapsto a(0,y)|dy|^s. \tag{4.48} $$

The subspace $\dot{\mathcal{C}}^\infty(X;E) \subset \mathcal{C}^\infty(X;E)$ given by (4.44) is closed. As a result there are two natural spaces of distributional half-densities:

$$ \begin{aligned} \mathcal{C}^{-\infty}(X; {}^b\Omega^{\frac12}) &= \left(\dot{\mathcal{C}}^\infty(X; {}^b\Omega^{\frac12})\right)' \quad \text{(extendible distributions)} \\ \dot{\mathcal{C}}^{-\infty}(X; {}^b\Omega^{\frac12}) &= \left(\mathcal{C}^\infty(X; {}^b\Omega^{\frac12})\right)' \quad \text{(supported distributions).} \end{aligned} \tag{4.49} $$

The second space includes distributions supported on the boundary, the second does not. For the most part, only the spaces of extendible distributions occur below. The pairing (4.46) gives an identification of $\mathcal{C}^\infty(X; {}^b\Omega^{\frac12})$

as a subspace of the space of extendible distributions, as in the boundaryless case:

$$\mathcal{C}^\infty(X;{}^b\Omega^{\frac12}) \ni \nu \longmapsto \nu' \in \mathcal{C}^{-\infty}(X;{}^b\Omega^{\frac12}),$$
$$\nu'(\phi) = \int_X \nu\phi,\ \phi \in \dot{\mathcal{C}}^\infty(X;{}^b\Omega^{\frac12}).$$

As usual there is rarely any point in distinguishing between a $\mathcal{C}^\infty$ density and the distribution it represents.

The corresponding construction for manifolds with corners is also needed to a limited extent. To avoid a detailed discussion it is enough to note that they behave locally as products of manifolds with boundary. Then spaces such as $\dot{\mathcal{C}}^\infty(X;E)$ can be defined by reference to any local product decomposition. For more details see [63]. For a general vector bundle E over a compact manifold with corners, X, define

$$\mathcal{C}^{-\infty}(X;E) = \left(\dot{\mathcal{C}}^\infty(X;E' \otimes {}^b\Omega)\right)'.$$

This reduces to the first case of (4.49) when $E = {}^b\Omega^{\frac12}$. The Schwartz kernel theorem extends readily to manifolds with boundary in terms of these spaces:

THEOREM 4.18. *(Schwartz) There is a 1-1 correspondence between continuous linear maps*

$$\dot{\mathcal{C}}^\infty(X;E) \longrightarrow \mathcal{C}^{-\infty}(Y;F), \tag{4.50}$$

where X and Y are compact manifolds with boundary and E and F are vector bundles over them, and the space $\mathcal{C}^{-\infty}(Y \times X; F \boxtimes [E^* \otimes {}^b\Omega])$; the correspondence is given by

$$\begin{gathered} A \longrightarrow K \Longleftrightarrow \langle A\phi, \psi\rangle = \langle K, \psi \boxtimes \phi\rangle \\ \forall\ \phi \in \dot{\mathcal{C}}^\infty(X;E), \psi \in \dot{\mathcal{C}}^\infty(Y;F^* \otimes {}^b\Omega). \end{gathered} \tag{4.51}$$

Actually this result is not really used below. Rather it is of philosophical importance, since it shows that questions about operators can always be reduced to questions about distributions. It is worth noting the general approach to understanding operators that underlies this discussion. As seen in Chapter 1, the kernels of operators related to b-differential operators tend to have rather complicated singularities at the corner of the manifold $X^2 = X \times X$. The stretched product X_b^2 has been introduced, since the kernels are simplified when lifted to it. Observe that the blow-down map has the property

$$(\beta_b^2)^* : \dot{\mathcal{C}}^\infty(X^2;{}^b\Omega^{\frac12}) \longleftrightarrow \dot{\mathcal{C}}^\infty(X_b^2;{}^b\Omega^{\frac12}). \tag{4.52}$$

EXERCISE 4.19. Check (4.52) in local coordinates.

Therefore, by duality,

$$(\beta_b^2)_* : \mathcal{C}^{-\infty}(X_b^2; {}^b\Omega^{\frac{1}{2}}) \longleftrightarrow \mathcal{C}^{-\infty}(X^2; {}^b\Omega^{\frac{1}{2}}). \tag{4.53}$$

Thus, nothing is gained by blowing up the corner! Of course that is not the point. The idea is that, by (4.53) and its extension to sections of other bundles, one can just as well examine the kernels of operators (4.50) on X_b^2 as on X^2. There are 'more' $\mathcal{C}^\infty$ functions on X_b^2, so more kernels will be considered as admissible than on X^2.

A useful combination of (4.53) and Theorem 4.18 in the special case that $X = Y$ is:

LEMMA 4.20. *If E and F are $\mathcal{C}^\infty$ vector bundles over a compact manifold with boundary X then the continuous linear operators*

$$\dot{\mathcal{C}}^\infty(X; E) \longrightarrow \mathcal{C}^{-\infty}(X; F)$$

are in 1-1 correspondence with the elements of the space of distributional sections $\mathcal{C}^{-\infty}(X_b^2; (\beta_b^2)^*(F \boxtimes [E' \otimes {}^b\Omega]))$, where

$$A \longleftrightarrow \kappa \Longleftrightarrow \langle A\phi, \psi\rangle = \langle \kappa, (\beta_b^2)^*[\psi \boxtimes \phi]\rangle. \tag{4.54}$$

In particular if $E = F = {}^b\Omega^{\frac{1}{2}}$, which will be assumed until §4.16, then $F \boxtimes [E' \otimes {}^b\Omega] \equiv {}^b\Omega^{\frac{1}{2}}(X^2)$. Moreover it follows from the discussion above that there is a natural isomorphism (*cf.* (4.26))

$$(\beta_b^2)^*[{}^b\Omega^{\frac{1}{2}}(X^2)] \equiv {}^b\Omega^{\frac{1}{2}}(X_b^2). \tag{4.55}$$

Thus, finally, operators on distributional densities can be identified with the lifts of their kernels to X_b^2 :

(4.56)
$$\left\{A_K : \dot{\mathcal{C}}^\infty(X; {}^b\Omega^{\frac{1}{2}}) \longrightarrow \mathcal{C}^{-\infty}(X; {}^b\Omega^{\frac{1}{2}})\right\} \longleftrightarrow \left\{K_A \in \mathcal{C}^{-\infty}(X_b^2; {}^b\Omega^{\frac{1}{2}})\right\}.$$

4.8. Kernels of b-differential operators.

Consider (4.56) applied to the identity operator on b-half-densities. The action of the identity is given in local coordinates by

$$\mathrm{Id}(\phi\left|\frac{dx}{x}dy\right|^{\frac{1}{2}}) = \int \delta(x-x')\delta(y-y')\phi(x',y')dx'dy'\left|\frac{dx}{x}dy\right|^{\frac{1}{2}}.$$

Thus the kernel in the sense of (4.51) is

$$K_{\mathrm{Id}} = x'\delta(x-x')\delta(y-y')\left|\frac{dx'}{x'}dy'\frac{dx}{x}dy\right|^{\frac{1}{2}}. \tag{4.57}$$

By a smooth Dirac section of order (at most) k of ${}^b\Omega^{\frac{1}{2}}$, with respect to Δ_b, is meant $\kappa \in \mathcal{C}^{-\infty}(X_b^2; {}^b\Omega^{\frac{1}{2}})$ which has support contained in Δ_b and in local coordinates takes the form

$$\sum_{p+|\alpha|\in k} a_{p,\alpha}(x,y) D_s^p \delta(s-1) D_y^\alpha \delta(y-y')$$

where the coefficients are $\mathcal{C}^\infty$. One expression of the utility of X_b^2 is

LEMMA 4.21. *Under the isomorphism* (4.56) *the space* $\mathrm{Diff}_b^k(X; {}^b\Omega^{\frac{1}{2}})$ *is mapped isomorphically onto the space of all smooth Dirac sections of order* k, *with respect to* Δ_b.

PROOF: Lifting to X_b^2, where only a neighbourhood of Δ_b needs to be considered, the coordinates x' and $x/x' = s$ can be used. Thus (4.57) becomes

$$\delta(s-1)\delta(y-y')\left|\frac{ds}{s} dy \frac{dx'}{x'} dy'\right|^{\frac{1}{2}}.$$

Writing $P = P \circ \mathrm{Id}$ for a general element of $\mathrm{Diff}_b^k(X; {}^b\Omega^{\frac{1}{2}})$ and using (4.28) it follows that the lift is a smooth Dirac section and conversely that all such sections arise in this way.

Thus the degeneracy of the kernels on X^2 is removed by the lift to X_b^2. The space of kernels is then much the same as in the boundaryless case and the process of microlocalization can be applied directly.

4.9. The small space of b-pseudodifferential operators.

After all this preparation the (small) space of b-pseudodifferential operators can now be defined.

DEFINITION 4.22. The (small) space, $\Psi_b^m(X; {}^b\Omega^{\frac{1}{2}})$, of b-pseudodifferential operators of order m, acting on b-half-densities, consists of those continuous linear operators which under (4.56) correspond to conormal sections of order m associated to the lifted diagonal and vanishing to all orders at $\mathrm{lb} \cup \mathrm{rb}$:

$$\Psi_b^m(X; {}^b\Omega^{\frac{1}{2}}) \longleftrightarrow \left\{\kappa \in I^m(X_b^2, \Delta_b; {}^b\Omega^{\frac{1}{2}}); \kappa \equiv 0 \text{ at } \mathrm{lb} \cup \mathrm{rb}\right\}. \tag{4.58}$$

Here $\equiv$ means equality of Taylor series at the indicated set.

Often (4.58) will be treated as an equality. Of course the precise meaning of the space on the right in (4.58) still needs to be explained. As noted in Lemma 4.1, the lifted diagonal is an embedded $\mathcal{C}^\infty$ submanifold, of dimension $\dim X$, which meets the boundary of X_b^2 only at the front face,

$\mathrm{bf}(X_b^2)$. Thus the vanishing conditions in (4.58) make sense since all elements of $I^m(X_b^2, \Delta_b; {}^b\Omega^{\frac{1}{2}})$ are $\mathcal{C}^\infty$ in a neighbourhood of $\mathrm{lb} \cup \mathrm{rb}$, this being disjoint from Δ_b.

So consider this definition in a little more detail. The meaning of (4.58) is that the kernel, κ, must have the following properties:

$$\kappa_{|X_b^2 \backslash \Delta_b} \in \mathcal{C}^\infty\left(X_b^2 \backslash \Delta_b; {}^b\Omega^{\frac{1}{2}}\right) \tag{4.59}$$
$$\kappa \equiv 0 \text{ at } \mathrm{lb} \sqcup \mathrm{rb}$$

$$\kappa(z, z') = (2\pi)^{-n-1} \int e^{i(z-z')\zeta} a(z', \zeta) d\zeta |dz dz'|^{\frac{1}{2}} \text{ near } \Delta_b \backslash \mathrm{bf} \tag{4.60}$$

$$\begin{aligned} \kappa(r, \tau, y, y') = (2\pi)^{-n-1} &\int e^{i\tau\lambda + i(y-y')\cdot\eta} b(r, y', \lambda, \eta) d\lambda d\eta \\ &\times \left| \frac{dr}{r} d\tau dy dy' \right|^{\frac{1}{2}} \text{ near } \Delta_b \cap \mathrm{bf}\,. \end{aligned} \tag{4.61}$$

In (4.60) $z_0, \dots, z_n$ and $z'_0, \dots, z'_n$ are the same coordinates in the two factors of $X^2 \simeq X_b^2$ near $\Delta_b \backslash \mathrm{bf} \simeq \Delta \backslash (\partial X)^2$ and in (4.61) if x, y, x', y' are the local coordinates with the usual conventions then $\tau = (x - x')/(x + x')$, $r = x + x'$. The amplitudes a and b are also required to be symbols of order m, as in (4.42). If the symbols have full asymptotic expansion as in (4.43) then the 'classical' or one-step polyhomogeneous class results:

$$\Psi^m_{b,\mathrm{os}}(X; {}^b\Omega^{\frac{1}{2}}) \longleftrightarrow \left\{ \kappa \in I^m_{\mathrm{os}}(X_b^2, \Delta_b; {}^b\Omega^{\frac{1}{2}}); \kappa \equiv 0 \text{ at } \mathrm{lb} \sqcup \mathrm{rb} \right\}. \tag{4.62}$$

For the most part the various properties of these conormal spaces (including coordinate-invariance) will be taken for granted, since that is precisely what is covered in a standard treatment of pseudodifferential operators (on manifolds without boundary). A little enlightenment may be gained from the exercises starting at Exercise 4.24 below. A safer alternative for the uninitiated is to consult [47, Chapter 18] or one of many reasonable introductions to pseudodifferential operators (for example [83], [88]). Another possibility is [65].

4.10. Symbol map.

One particular property of conormal spaces is that they have symbol mappings delineating their order filtrations. Let $S^{[m]}({}^bT^*X)$ denote the space of $\mathcal{C}^\infty$ functions, on ${}^bTX \backslash 0$, which are homogeneous of degree m. Summarizing the invariance discussion in the exercises leads to:

PROPOSITION 4.23. *The local symbols in* (4.60), (4.61) *together fix the symbol map, giving a short exact sequence*

$$(4.63)\quad 0 \longrightarrow \Psi^{m-1}_{b,\mathrm{os}}(X;{}^b\Omega^{\frac12}) \hookrightarrow \Psi^{m}_{b,\mathrm{os}}(X;{}^b\Omega^{\frac12}) \xrightarrow{{}^b\sigma_m} S^{[m]}({}^bT^*X) \longrightarrow 0.$$

EXERCISE 4.24. Check that if κ is defined by (4.60) or (4.61), where a or b satisfies the estimates of (4.42), then it satisfies (4.59). [Hint: These integrals are inverse Fourier transforms. Show that if $m < -q - N$ for the order in (4.42) then κ is $\mathcal{C}^N$. Then use the standard identities for the Fourier transform of $\tau\kappa$ and $(y-y')_j\kappa$ to show that if $|\tau|^2 + |y-y'|^2 \neq 0$ then κ is in $\mathcal{C}^N$ for each N.]

EXERCISE 4.25. Use Exercise 4.24 to show that if $U \subset X^2_b$ is a preassigned neighbourhood of Δ_b and κ satisfies (4.59) – (4.61) for some covering of Δ_b by coordinate patches then $\kappa = \kappa_1 + \kappa_2$, where $\kappa_1 \in \mathcal{C}^\infty(X^2_b;{}^b\Omega^{\frac12})$ satisfies (4.59) and κ_2 has support in U and still satisfies (4.59) – (4.61) with respect to the same coordinate covering.

EXERCISE 4.26. Use Exercise 4.25 to show that κ satisfies (4.59) – (4.61) for a particular covering by coordinate patches if and only if $\kappa = \kappa' + \kappa''$, where κ'' is $\mathcal{C}^\infty$ and κ' is a sum of terms, each supported in one coordinate patch and satisfying (4.60) or (4.61) in those coordinates.

EXERCISE 4.27. For distributions supported in the coordinate patch check directly that (4.48) and (4.61) are invariant under changes of coordinates on X^2_b which are linear in $z - z'$ or $y - y'$ and τ but arbitrary in z' or r and y'. Show that under such transformations the symbol a or b projects to a well-defined element of $S^{[m]}({}^bT^*X)$, i.e. the leading part is unchanged.

EXERCISE 4.28. Finally (this is the trickiest part) show that under coordinate transformations which induce a trivial transformation on $N\Delta_b$ the form of (4.60) or (4.61) is preserved and the symbol is fixed in $S^{[m]}({}^bT^*X)$. Now check that you understand where (4.63) comes from.

The utility of the algebra of pseudodifferential operators on a compact manifold without boundary is closely linked to the symbol homomorphism. This is a map into a commutative algebra (really a *family* of maps into a *family* of algebras) which determines the leading part of an operator. Using this symbol map one can readily construct parametrices for elliptic operators. For the (small) calculus of b-pseudodifferential operators there are *two* such "normal homomorphisms." The first, in (4.63), is a direct extension of the standard symbol map and has similar properties. The second, indicial, homomorphism discussed below, takes values in a non-commutative algebra and is therefore somewhat different. Both homomorphisms are involved in the construction of parametrices.

4.11. Elementary mapping properties.

By (4.53) b-pseudodifferential operators do indeed correspond to linear operators

$$\dot{\mathcal{C}}^\infty(X; {}^b\Omega^{\frac12}) \longrightarrow \mathcal{C}^{-\infty}(X; {}^b\Omega^{\frac12}), \tag{4.64}$$

but they have much better properties than this. To exploit the properties of the kernel it is useful to have a representation of its action, as in (4.54), which uses the lift of the kernel to X_b^2. The operator can be written out in terms of its kernel on X^2

$$A\phi(z) = \int_X K(z, z')\phi(z'). \tag{4.65}$$

Suppose that $\phi \in \dot{\mathcal{C}}^\infty(X; {}^b\Omega^{\frac12})$ has support in a small, product, neighbourhood of the boundary where x is a defining function for the boundary. Then (4.65) can be written

$$A\phi(x, y) = \int_0^\epsilon \int_{\partial X} K(x, y, x', y')\phi(x', y')\frac{dx'}{x'}dy'|\frac{dx}{x}dy|^{\frac12},\ x < \epsilon, \tag{4.66}$$

where the density factors have also be written explicitly.

In (4.66) only the properties of the kernel near $x = x' = 0$ are involved. For simplicity assume that

$$\operatorname{supp}(K) \subset [0, \epsilon] \times \partial X \times [0, \epsilon] \times \partial X.$$

Then the coordinate τ from (4.16) can be used to lift from X^2 to X_b^2 :

$$A\phi(x, y) = \int_{-1}^1 \int_{\partial X} K(x, y, x\frac{1-\tau}{1+\tau}, y')\phi(x\frac{1-\tau}{1+\tau}, y')\frac{d\tau}{1-\tau^2}dy'|\frac{dx}{x}dy|^{\frac12}.$$

Since the lift of the kernel to X_b^2 is by definition

$$\kappa(r, \tau, y, y')|\frac{d\tau}{1-\tau^2}drdydy'|^{\frac12} = K(x, y, x\frac{1-\tau}{1+\tau}, y')|\frac{dx}{x}dy\frac{dx'}{x'}dy'|^{\frac12},$$

this can be written as

$$A\phi(x, y) = \int_{-1}^1 \int_{\partial X} \kappa(\frac{2x}{1+\tau}, \tau, y, y')\phi(x\frac{1-\tau}{1+\tau}, y')\frac{d\tau}{1-\tau^2}dy'|\frac{dx}{x}dy|^{\frac12}. \tag{4.67}$$

In the form (4.67) it can be seen directly how the properties of the kernel demanded in (4.58) will be reflected in the properties of the operators. However it is usually clearer to look at the kernel as a function of the improper variables s, x or t, x', where s is given by (4.10):

$$s = \frac{1+\tau}{1-\tau} = \frac{x}{x'} \text{ and } t = \frac{1}{s}.$$

The coordinates s, x fail near rb and $\mathrm{lb} \cap \mathrm{bf}$ since $s = \infty$ at rb and at $\mathrm{lb} \cap \mathrm{bf}$ $x = s\rho_{\mathrm{lb}}$, where ρ_{lb} is a defining function for $\mathrm{lb}(X_b^2)$, has vanishing differential. On the other hand t, x' fail as coordinates at lb and $\mathrm{rb} \cap \mathrm{bf}$. (Of course in all cases local coordinates y, y' in the two factors of ∂X are also needed.) However, by definition in (4.58) the kernel κ is $\mathcal{C}^\infty$ and vanishes rapidly at $s = 0$ and $s = \infty$, which are just lb and rb. Thus s can be used in place of τ in (4.67) without having to worry about convergence problems. This leads to the two useful representations:

$$A\phi(x, y) = \int_0^\infty \int_{\partial X} \kappa'(x, s, y, y')\phi(x/s, y')\frac{ds}{s}dy'|\frac{dx}{x}dy|^{\frac{1}{2}}, \tag{4.68}$$

$$A\phi(x, y) = \int_0^\infty \int_{\partial X} \kappa''(x/s, s, y, y')\phi(x/s, y')\frac{ds}{s}dy'|\frac{dx}{x}dy|^{\frac{1}{2}}, \tag{4.69}$$

where κ' stands for the kernel expressed in terms of x, s, y, y', κ'' for the kernel expressed in terms of x', s, y, y' and $\phi \in \dot{\mathcal{C}}_c^\infty([0, \epsilon) \times \partial X; {}^b\Omega^{\frac{1}{2}})$. These representations will be used immediately to improve (4.64):

PROPOSITION 4.29. *Each $A \in \Psi_b^m(X; {}^b\Omega^{\frac{1}{2}})$ defines a linear operator*

$$A : \dot{\mathcal{C}}^\infty(X; {}^b\Omega^{\frac{1}{2}}) \longrightarrow \dot{\mathcal{C}}^\infty(X; {}^b\Omega^{\frac{1}{2}}). \tag{4.70}$$

PROOF: Divide the kernel of the operator into three pieces, using a $\mathcal{C}^\infty$ partition of unity on X^2. Namely

$$A = A_1 + A_2 + A_3, \tag{4.71}$$

where A_3 has kernel supported in $x \geq \epsilon$, A_2 has kernel supported in $x \leq 2\epsilon$, $x' > x \geq \epsilon$, and A_1 has kernel supported in $x, x' \leq 2\epsilon$.

Of the pieces, A_2 is the most innocuous, since the support of its kernel does not touch the diagonal in X^2 and hence it is $\mathcal{C}^\infty$ on X^2 and vanishes to infinite order at both boundaries. It is convenient to have a name for such operators:

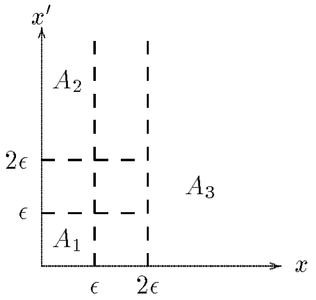

Figure 6. Decomposition in (4.69).

DEFINITION 4.30. The *maximally residual operators,* in $\Psi^{-\infty,\emptyset}(X;{}^b\Omega^{\frac12})$, are those operators with kernels in $\dot{\mathcal{C}}^\infty(X^2;{}^b\Omega^{\frac12}) = \dot{\mathcal{C}}^\infty(X^2_b;{}^b\Omega^{\frac12})$.

The notation here is a precursor to that for the full calculus in Chapter 5. The $-\infty$ refers to the absence of any singularity at the diagonal, while the $\emptyset$ means that the kernel vanishes to infinite order at all boundaries. The absence of the, otherwise ubiquitous, subscript 'b' is also no accident; it refers to the fact that these kernels are characterized directly on X^2.

Essentially from the definition of the action of an operator in terms of its Schwartz kernel and the spaces involved

$$A \in \Psi^{-\infty,\emptyset}(X;{}^b\Omega^{\frac12}) \Longleftrightarrow A\colon \mathcal{C}^{-\infty}(X;{}^b\Omega^{\frac12}) \longrightarrow \dot{\mathcal{C}}^\infty(X;{}^b\Omega^{\frac12}). \tag{4.72}$$

EXERCISE 4.31. Make sure you understand why the mapping property (4.72) is a necessary and sufficient condition for an operator to be maximally residual.

Certainly then A_2 satisfies (4.70). Slightly more seriously consider A_3. Removing another maximally residual term the kernel of A_3 can be supposed to have support contained strictly in the interior. From the definition A_3 is then simply a pseudodifferential operator, in the usual sense, on the interior $\overset{\circ}{X}$ of X with kernel having compact support in $(\overset{\circ}{X})^2$. One of the standard mapping properties of pseudodifferential operators is that they preserve $\mathcal{C}^\infty$ and taking into account the support property this means

$$A_3\colon \mathcal{C}^\infty(X;{}^b\Omega^{\frac12}) \longrightarrow \dot{\mathcal{C}}^\infty(X;{}^b\Omega^{\frac12}). \tag{4.73}$$

Recall where this mapping property comes from. It is really a result on the regularity of push-forwards of conormal distributions. More prosaically it follows from (and is essentially equivalent to) the fact that if K is a kernel as in (4.60) (with compact support) then

$$\int K(z,z')dz' \in \mathcal{C}^\infty. \tag{4.74}$$

EXERCISE 4.32. Show that (4.74) follows from (4.60), the properties of symbols discussed in the exercises starting at Exercise 4.24 and integration by parts.

So finally consider A_1. Here the representation (4.68) can be used. The kernel κ vanishes to all orders at $s = 0$ and rapidly as $s \longrightarrow \infty$, so it follows that the product

$$\begin{gathered} \kappa(x,s,y,y')\phi(x/s,y') \text{ is } \mathcal{C}^\infty \text{ in } |s-1| \neq 0 \\ \text{and vanishes rapidly as } s \longrightarrow \infty \text{ and as } s \downarrow 0. \end{gathered} \tag{4.75}$$

Indeed, when $s \neq 1$, κ itself is $\mathcal{C}^\infty$ and if $s > 0$ then so is $\phi(x/s, y')$. Since, by assumption, κ vanishes rapidly as $s \downarrow 0$ so does the product; the derivatives of $\phi(x/s,y')$ being of at most polynomial growth as $s \downarrow 0$. As $s \longrightarrow \infty$ both factors vanish rapidly (since x is bounded). This rapid decrease at infinity compensates for the non-compactness of the domain of integration, so the integral of (4.75) in s and y' behaves just as for (4.74). Thus $A_1\phi$ is $\mathcal{C}^\infty$. Since $\phi(x',y')$ is assumed to vanish with all derivatives at $x' = 0$, i.e. $\phi(x',y') = (x')^N\phi_N(x',y')$ the same argument can be applied to $\kappa(x,s,y,y')s^{-N}\phi_N(x/s,y')$ and hence it follows that $A_1\phi \in x^N\mathcal{C}^\infty(X;{}^b\Omega^{\frac12})$ for any N. This proves (4.70) and therefore Proposition 29

EXERCISE 4.33. Using a (simple) duality argument show that

$$A \in \Psi_b^m(X;{}^b\Omega^{\frac12}) \Longrightarrow A\colon \mathcal{C}^{-\infty}(X;{}^b\Omega^{\frac12}) \longrightarrow \mathcal{C}^{-\infty}(X;{}^b\Omega^{\frac12}) \tag{4.76}$$

is a continuous linear operator.

Essentially the same argument as used to prove Proposition 4.29 also leads to

PROPOSITION 4.34. *Each element $A \in \Psi_b^*(X;{}^b\Omega^{\frac12})$ defines a continuous linear map*

$$A\colon \mathcal{C}^\infty(X;{}^b\Omega^{\frac12}) \longrightarrow \mathcal{C}^\infty(X;{}^b\Omega^{\frac12}). \tag{4.77}$$

PROOF: Using the same decomposition (4.71) both A_2 and A_3 have the property (4.77) as consequences of (4.72) and (4.73). So again it is enough to consider A_1. Notice that $\dot{\mathcal{C}}^\infty(X;{}^b\Omega^{\frac12})$ is not dense in $\mathcal{C}^\infty(X;{}^b\Omega^{\frac12})$ in the topology of the latter, but it is dense in the topology of $\mathcal{C}^{-\infty}(X;{}^b\Omega^{\frac12})$. Thus the definition of the map in (4.77) follows from (4.76). Explicitly this convergence follows by taking

$$\phi_n(x,y)=\mu(nx)\phi(x,y),\ \mu\in\mathcal{C}^\infty([0,\infty)),\begin{cases}\mu(t) & =1,\ t>1\\ \mu(t) & =0,\ t<\frac12.\end{cases}$$

Inserting ϕ_n into (4.68) the integral remains, for $x>0$, absolutely convergent as $n\longrightarrow\infty$. Thus the remainder of the proof of Proposition 4.29 can be followed since the rapid vanishing of ϕ is only used at the very last step. Thus (4.77) holds.

4.12. Asymptotic completeness.

A companion property to the symbol map in (4.63) is the *asymptotic completeness* of the space. Define

$$\Psi_b^{-\infty}(X;{}^b\Omega^{\frac12})=\bigcap_m\Psi_b^m(X;{}^b\Omega^{\frac12}).$$

Then, under (4.62), there is an identification

$$\Psi_b^{-\infty}(X;{}^b\Omega^{\frac12})\longleftrightarrow\left\{\kappa\in\mathcal{C}^\infty(X_b^2;{}^b\Omega^{\frac12});\kappa\equiv0\text{ at }\mathrm{lb}\sqcup\mathrm{rb}\right\}.$$

EXERCISE 4.35. Show that, for any $m\in\mathbb{R}$,

$$\Psi_b^{-\infty}(X;{}^b\Omega^{\frac12})=\bigcap_j\Psi_{b,\mathrm{os}}^{m-j}(X;{}^b\Omega^{\frac12}).$$

Give an example to make sure you understand that this space is *a lot bigger* than $\Psi^{-\infty,\emptyset}(X;{}^b\Omega^{\frac12})$.

Asymptotic completeness is a general property of conormal spaces. For the one-step polyhomogeneous spaces it becomes:

LEMMA 4.36. *If* $A_j\in\Psi_{b,\mathrm{os}}^{m-j}(X;{}^b\Omega^{\frac12})$, $j=0,1,\dots$ *then there exists an asymptotic sum* $A\in\Psi_{b,\mathrm{os}}^m(X;{}^b\Omega^{\frac12})$ *such that*

$$A-\sum_{j=0}^{N-1}A_j\in\Psi_{b,\mathrm{os}}^{m-N}(X;{}^b\Omega^{\frac12})\ \forall\ N. \tag{4.78}$$

The relationship (4.78) is written

$$A\sim\sum_{j=0}^{\infty}A_j$$

and determines A uniquely up to an element of $\Psi_b^{-\infty}(X;{}^b\Omega^{\frac12})$.

4.13. Small parametrix.

Of course there is one other important property that guided the definition of the small calculus from the beginning. Namely

$$\mathrm{Diff}_b^k(X;{}^b\Omega^{\frac12}) \subset \Psi_{b,\mathrm{os}}^k(X;{}^b\Omega^{\frac12})\ \forall\ k \in \mathbb{N}_0. \tag{4.79}$$

This will be used with the following special case of a more general composition formula (Proposition 5.20) proved later:

LEMMA 4.37. *For any $k \in \mathbb{N}$, $m \in \mathbb{R}$*

$$\begin{gathered} \mathrm{Diff}_b^k(X;{}^b\Omega^{\frac12}) \circ \Psi_{b,\mathrm{os}}^m(X;{}^b\Omega^{\frac12}) \subset \Psi_{b,\mathrm{os}}^{m+k}(X;{}^b\Omega^{\frac12}) \\ {}^b\sigma_{m+k}(P\circ A) = {}^b\sigma_k(P)\cdot{}^b\sigma_m(A), \end{gathered} \tag{4.80}$$

where ${}^b\sigma_k(P)$ is the symbol defined in (2.23) and ${}^b\sigma_m(A)$ is from (4.63). Since all these spaces are invariant under passage to adjoints, the same result holds for the composite in the other order.

PROOF: To prove (4.79) the first thing to check is that

$$\mathrm{Id} \in \Psi_{b,\mathrm{os}}^0(X;{}^b\Omega^{\frac12}). \tag{4.81}$$

This however is a direct consequence of Lemma 4.21.

Now (4.79) can be deduced from (4.81) by trivializing the density bundles and recalling (2.20). Thus it is only necessary to show that if $\nu_b^{\frac12}$ is a non-vanishing b-half-density

$$\Psi_{b,\mathrm{os}}^m(X;{}^b\Omega^{\frac12}) \ni \kappa \longmapsto \nu_b^{\frac12}V\nu_b^{-\frac12}\cdot\kappa \in \Psi_{b,\mathrm{os}}^{m+1}(X;{}^b\Omega^{\frac12})\ \forall\ V \in \mathcal{V}_b(X). \tag{4.82}$$

Here V acts on the left. The lift of these vector fields to X_b^2 is described by Proposition 4.4. Since the lifted vector field is $\mathcal{C}^\infty$ on X_b^2 and tangent to the boundary, (4.82) holds, as can be seen directly from the local representations of the kernels in (4.61). From this (4.79) and (4.80) follow.

These elementary properties of the small calculus allow the first parametrix construction to be made. Really a parametrix should be an inverse up to compact errors; this is *not* achieved yet, as discussed below. However it is worth stretching the concept a little and thinking of this as a parametrix construction:

PROPOSITION 4.38. *If $P \in \mathrm{Diff}_b^k(X;{}^b\Omega^{\frac12})$ is elliptic then there exists $G_s \in \Psi_{b,\mathrm{os}}^{-k}(X;{}^b\Omega^{\frac12})$ such that*

$$R_s = \mathrm{Id} - P\circ G_s \in \Psi_b^{-\infty}(X;{}^b\Omega^{\frac12}), \tag{4.83}$$

and G_s with this property is unique up to an element of $\Psi_b^{-\infty}(X;{}^b\Omega^{\frac12})$.

The subscript 's' is intended to indicate that the element is in the small calculus.

Proof: This is just *the* standard symbolic construction. The ellipticity of P means that there exists

$$q \in S^{-k}({}^bT^*X) \text{ with } q \cdot {}^b\sigma_k(P) - 1 \in S^{-\infty}({}^bT^*X).$$

Indeed q can be taken to be equal to $1/{}^b\sigma_k(P)$ outside any compact neighbourhood of the zero section of ${}^bT^*X$. Then the surjectivity in (4.63) shows that there exists

$$G_s^0 \in \Psi_{b,\mathrm{os}}^{-k}(X; {}^b\Omega^{\frac12}) \text{ with } {}^b\sigma_{-k}(G_s^0) = q.$$

From (4.80) it follows that $P \circ G_s^0 \in \Psi_{b,\mathrm{os}}^0(X; {}^b\Omega^{\frac12})$ and

$${}^b\sigma_0(P \circ G_s^0) = 1.$$

Certainly ${}^b\sigma_0(\mathrm{Id}) = 1$, so ${}^b\sigma_0(R_s^1) = 0$ if

$$R_s^1 = \mathrm{Id} - P \circ G_s^0. \tag{4.84}$$

From the exactness in (4.63), $R_s^1 \in \Psi_{b,\mathrm{os}}^{-1}(X; {}^b\Omega^{\frac12})$. Moreover G_s^0 is determined up to the addition of an element of $\Psi_{b,\mathrm{os}}^{-k-1}(X; {}^b\Omega^{\frac12})$. Now proceed by induction. Suppose that

$$G_s^{(j)} \in \Psi_{b,\mathrm{os}}^{-k-j}(X; {}^b\Omega^{\frac12}) \text{ is chosen for } j < l \tag{4.85}$$

so that

$$R_s^l = \mathrm{Id} - P \circ \left(\sum_{j=0}^{l-1} G_s^{(j)} \right) \in \Psi_{b,\mathrm{os}}^{-l}(X; {}^b\Omega^{\frac12}). \tag{4.86}$$

Then (4.63) can be used again to find $G_s^{(l)} \in \Psi_{b,\mathrm{os}}^{-k-l}(X; {}^b\Omega^{\frac12})$ with

$${}^b\sigma_{-k-l}(G_s^{(l)}) = q \cdot {}^b\sigma_{-l}(R_s^l).$$

This means $R_s^l - P \circ G_s^{(l)} \in \Psi_{b,\mathrm{os}}^{-l-1}(X; {}^b\Omega^{\frac12})$, so the inductive hypothesis (4.84), (4.85) is recovered.

Finally take

$$G_s \sim \sum_{j=0}^{\infty} G_s^{(j)} \in \Psi_{b,\mathrm{os}}^{-k}(X; {}^b\Omega^{\frac12}).$$

Then from (4.78) and (4.86)

$$R_s = \mathrm{Id} - P \circ G_s \in \Psi_b^{-\infty}(X; {}^b\Omega^{\frac{1}{2}});$$

this proves (4.83). The essential uniqueness follows from the fact that if $G'_s \in \Psi_b^{-k}(X; {}^b\Omega^{\frac{1}{2}})$ is another solution then

$$P \circ (G_s - G'_s) \in \Psi_b^{-\infty}(X; {}^b\Omega^{\frac{1}{2}}).$$

Proceeding inductively shows that $G_s - G'_s \in \Psi_b^{-k-j}(X; {}^b\Omega^{\frac{1}{2}})$ for all j, i.e. $G_s - G'_s \in \Psi_b^{-\infty}(X; {}^b\Omega^{\frac{1}{2}})$.

A solution to

$$R'_s = \mathrm{Id} - \tilde{G}_s \circ P \in \Psi_b^{-\infty}(X; {}^b\Omega^{\frac{1}{2}}) \tag{4.87}$$

can be constructed by a similar argument. Moreover $\tilde{G}_s$, constructed to satisfy (4.87), is essentially equal to G_s. To show this requires a particular case of Proposition 5.20, where m' or $m = -\infty$. Then,

$$\tilde{G}_s - \tilde{G}_s \circ R_s = \tilde{G}_s \circ P \circ G_s = G_s - R'_s \circ G_s.$$

EXERCISE 4.39. Show, directly from the definition, that $\Psi_b^m(X; {}^b\Omega^{\frac{1}{2}})$ is invariant under passage to adjoints. Use this to show that a left parametrix, as in (4.87), can be obtained from a right parametrix for the adjoint of P.

4.14. Non-compactness.

As already noted, the error terms in (4.83) or (4.87) are not really very 'small'. In particular a general element of $\Psi_b^{-\infty}(X; {}^b\Omega^{\frac{1}{2}})$ is *not* compact on $L^2(X; {}^b\Omega^{\frac{1}{2}})$ (so far it has not even been shown to be bounded, but it is). To see why it is not compact, consider, for simplicity, the one-dimensional case. Then if $u = \phi(x)|dx/x|^{\frac{1}{2}}$ has support near $x = 0$ in $X = [0, 1]$,

$$Au(x) = \int_0^\infty \kappa(x, s)\phi(\frac{x}{s})\frac{ds}{s}|\frac{dx}{x}|^{\frac{1}{2}}. \tag{4.88}$$

Here $s = x/x'$ and κ can be assumed to have support in $[0, \frac{1}{2}] \times [\frac{1}{2}, 2]$, i.e $0 \le x \le \frac{1}{2}$, $\frac{1}{2} \le s \le 2$.

Now in (4.88) consider a sequence which converges weakly in L^2 :

$$u_t = \phi(\frac{x}{t})|\frac{dx}{x}|^{\frac{1}{2}} \quad \text{as } t \downarrow 0 \ u_t \rightharpoonup 0.$$

From (4.88) it follows that $Au_t = w_t(\frac{x}{t})|dx/x|^{\frac{1}{2}}$ if

$$w_t(x) = \int_0^\infty \kappa(xt, s)\phi(\frac{x}{s})\frac{ds}{s}.$$

Thus, as $t \downarrow 0$, w_t converges in $\mathcal{C}^\infty$ to

$$w_0(x) = \int_0^\infty \kappa(0, s)\phi(\frac{x}{s})\frac{ds}{s}.$$

It is straightforward to check that if $w_t \longrightarrow w_0$ in $\mathcal{C}^\infty([0, \frac{1}{2}])$ then $w_t(\frac{x}{t}) = Au_t \rightharpoonup 0$ (converges weakly to 0) in $L^2(X; {}^b\Omega^{\frac{1}{2}})$ but *cannot* converge strongly to zero unless $w_0 \equiv 0$.

EXERCISE 4.40. Show that if A is compact on $L^2(X; {}^b\Omega^{\frac{1}{2}})$ then $\kappa(0, s) \equiv 0$.

4.15. Indicial operator.

Thus for an element of $\Psi_b^{-\infty}(X; {}^b\Omega^{\frac{1}{2}})$ the *whole* of $\kappa \restriction \mathrm{bf}(X_b^2)$ appears to be (and is) an obstruction to compactness. This is not completely surprising since, under β_b, $\mathrm{bf}(X_b^2)$ gets mapped into $B \subset \Delta$, so the kernel on X^2 is singular at B if $\kappa \restriction \mathrm{bf} \neq 0$.

This suggests that this *obstruction* be looked at directly. To do so consider the map

$$\Psi_b^m(X; {}^b\Omega^{\frac{1}{2}}) \ni A \longmapsto \kappa_{\restriction \mathrm{bf}(X_b^2)} \in I^{m+\frac{1}{4}}(\mathrm{bf}, \Delta_{b,0}; {}^b\Omega^{\frac{1}{2}}). \tag{4.89}$$

Here $\Delta_{b,0} = \Delta_b \cap \mathrm{bf} = \{r = 0, \tau = 0, y = y'\}$, and ${}^b\Omega^{\frac{1}{2}}$ is the restriction to $\mathrm{bf}(X_b^2)$ of ${}^b\Omega^{\frac{1}{2}}(X_b^2)$ (see (4.47)). In fact there is a canonical identification

$${}^b\Omega^{\frac{1}{2}}(\mathrm{bf}) \simeq {}^b\Omega^{\frac{1}{2}}(X_b^2)_{\restriction \mathrm{bf}(X_b^2)}. \tag{4.90}$$

EXERCISE 4.41. If r, τ, y, y' are the usual type of coordinates in X_b^2 near bf, so $\mathrm{bf} = \{r = 0\}$, show that

$$\left|\frac{d\tau}{1-\tau^2}\frac{dr}{r}dydy'\right| \longmapsto \left|\frac{d\tau}{1-\tau^2}dydy'\right|$$

gives the canonical isomorphism

$${}^b\Omega(X_b^2)_{\restriction \mathrm{bf}(X_b^2)} \simeq {}^b\Omega(\mathrm{bf}(X_b^2))$$

and hence also gives (4.90).

The apparent increase in order in (4.89), from m to $m + \frac{1}{4}$, is a figment of the convention for the order of a conormal distribution, in particular of its relation to the dimension. This convention, introduced by Hörmander in [45], may seem a little strange but it is very natural in terms of pull-back and push-forward operations.

Recall, from (4.13), exactly what the front face is. A more primitive version of (4.4) can be used to define a compactified normal bundle to ∂X itself (rather than the spherical normal bundle to $B = \partial X \times \partial X$ in X^2). If $p \in \partial X$ then

$$ {}_+N_p(\partial X) = \{v \in T_pX; dx(v) \geq 0\}/T_p\partial X \tag{4.91}$$

is a closed half-line, forming a bundle over ∂X. Since it is reassuring to keep things, compact consider instead the product $X \times [0, \infty) \supset \partial X \times \{0\}$ and the space

$$ \widetilde{X}_p = [(\{v \in T_pX; dx(v) \geq 0\} \times [0, \infty)) \backslash \{0\}] / (T_p\partial X \times \mathbb{R}^+), \tag{4.92}$$

where the factor of $\mathbb{R}^+$ acts multiplicatively on T_pX.

This is very like (4.91), indeed

$$ {}_+N_p\partial X \ni [v] \hookrightarrow [(v, 1)] \in \widetilde{X}_p \tag{4.93}$$

embeds ${}_+N_p\partial X$ in $\widetilde{X}_p$. Again it follows that

$$\widetilde{X} = \bigsqcup_{p \in \partial X} \widetilde{X}_p$$

is a compact manifold with boundary, diffeomorphic to $\partial X \times [-1, 1]$ with a projective structure on its fibres; it is a natural compactification of ${}_+N\partial X$. It is useful to think of $\widetilde{X}$ as a *model* for X near ∂X. It has a natural $\mathbb{R}^+$-action, induced by the multiplicative action on the fibres at ${}_+N\partial X$. In (4.92) the $\mathbb{R}^+$-action is therefore

$$ [(v, 1)] \longrightarrow [(sv, 1)] \quad s > 0. \tag{4.94}$$

The two pieces of the boundary will be denoted

$$\partial\widetilde{X} \simeq \partial X \times \{-1\} \sqcup \partial X \times \{1\} = (\partial_0 X) \sqcup (\partial_1 X).$$

The stretching construction described above can be carried out, leading to the model stretched product $\widetilde{X}_b^2 = (\widetilde{X})_b^2$.

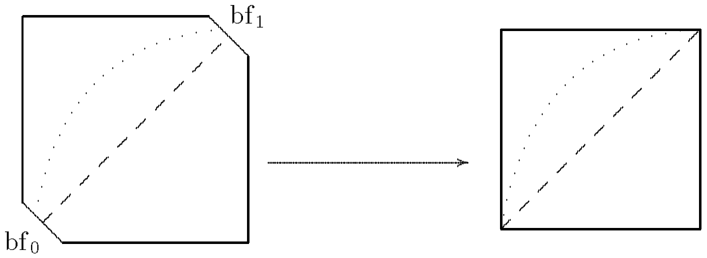

Figure 7. The blow-down map $\beta_b\colon \widetilde{X}^2_b \longrightarrow \widetilde{X}^2$ (with $\mathbb{R}^+$-orbit).

LEMMA 4.42. *The front face of $\widetilde{X}^2_b$ has two pieces,*

$$\mathrm{bf}(\widetilde{X}^2_b) = \mathrm{bf}_0(X^2_b) \sqcup \mathrm{bf}_1(X^2_b), \tag{4.95}$$

which are each canonically isomorphic to $\mathrm{bf}(X^2_b)$.

PROOF: First observe that (4.93) gives a canonical identification of one part of $\mathrm{bf}(\widetilde{X}^2_b)$ with $\mathrm{bf}([{}_+N\partial X]^2_b)$. Thus it is only necessary to show that

$$\mathrm{bf}(X^2_b) \cong \mathrm{bf}([{}_+N\partial X]^2_b).$$

This in turn is clear from (4.13).

As already noted there is a natural $\mathbb{R}^+$-action, M_a, on $\widetilde{X}$, coming from the multiplicative action on the fibres of ${}_+N\partial X$. This is generated by a naturally defined vector field, which will be denoted

$$x\frac{\partial}{\partial x} \in \mathcal{V}_b(\widetilde{X}).$$

The product $\widetilde{X}^2$ has the product action which, being generated by $x\partial/\partial x + x'\partial/\partial x'$, lifts to a $\mathcal{C}^\infty$ $\mathbb{R}^+$-action, M^2_a, on $\widetilde{X}^2_b$. Observe that for an operator the equivalence of the commutation property and $\mathbb{R}^+$-invariance:

$$M^*_a(Au) = A(M^*_a u) \Longleftrightarrow (M^2_a)^*\kappa = \kappa\ \forall\ a \in \mathbb{R}^+. \tag{4.96}$$

Thus $\widetilde{X}^2$ can be pictured as a square, suppressing the extra factor of $(\partial X)^2$. The action of M^2_a has orbits passing from one of the 'diagonal' corners to the other. Then $\widetilde{X}^2_b$ is the square with the two diagonal corners 'cut off,' i.e. blown up. The action of M^2_a is transversal to the front faces produced. Notice that, by assumption, the kernels of elements of the small calculus vanish to infinite order at boundary hypersurfaces other than the two front faces. Let $\Psi^m_{b,I,\mathrm{os}}(\widetilde{X};{}^b\Omega^{\frac12}) \subset \Psi^m_{b,\mathrm{os}}(\widetilde{X};{}^b\Omega^{\frac12})$ be the subspace of invariant elements in the sense of (4.96).

PROPOSITION 4.43. *Restriction to the first piece of the front face in* (4.95), *as in* (4.89), *gives an isomorphism:*

$$\Psi^m_{b,I,\mathrm{os}}(\widetilde{X};{}^b\Omega^{\frac12}) \longrightarrow \left\{\kappa \in I^{m+\frac14}_{\mathrm{os}}(\mathrm{bf}_0,\Delta^0_b;{}^b\Omega^{\frac12});\kappa\equiv 0 \text{ at } \mathrm{lb}\sqcup\mathrm{rb}\right\}. \tag{4.97}$$

Since the image space in (4.97) is exactly the same as for the restriction map (4.89) (on one-step polyhomogeneous operators) this defines the second 'normal homomorphism' which will be called the indicial homomorphism. Notice that the null space of (4.89) is exactly the space of kernels of the form $\rho_{\mathrm{bf}}\kappa$, where $\kappa\in\Psi^m_b(X;{}^b\Omega^{\frac12})$ and ρ_{bf} is a defining function for $\mathrm{bf}(X^2_b)$. Similarly for one-step polyhomogeneous operators:

PROPOSITION 4.44. *The restriction map* (4.89) *defines, using* (4.97), *a map*

$$\Psi^m_{b,\mathrm{os}}(X;{}^b\Omega^{\frac12})\ni A\longmapsto I(A)\in\Psi^m_{b,I,\mathrm{os}}(\widetilde{X};{}^b\Omega^{\frac12}) \tag{4.98}$$

which gives a short exact sequence

$$\begin{aligned}0\longrightarrow\rho_{\mathrm{bf}}\Psi^m_{b,\mathrm{os}}(X;{}^b\Omega^{\frac12})\hookrightarrow&\Psi^m_{b,\mathrm{os}}(X;{}^b\Omega^{\frac12})\\&\xrightarrow{I}\!\!\!\!\rightarrow\Psi^m_{b,I,\mathrm{os}}(\widetilde{X};{}^b\Omega^{\frac12})\longrightarrow 0\end{aligned} \tag{4.99}$$

where the first space in (4.99) *is fixed by*

$$\begin{gathered}\rho^k_{\mathrm{bf}}\Psi^m_{b,\mathrm{os}}(X;{}^b\Omega^{\frac12})=\\ \left\{\kappa\in\Psi^m_{b,\mathrm{os}}(X;{}^b\Omega^{\frac12});\kappa=\rho^k_{\mathrm{bf}}G,\ G\in\Psi^m_{b,\mathrm{os}}(X;{}^b\Omega^{\frac12})\right\}.\end{gathered}$$

The indicial operator $I(P)$ of $P\in\Psi^m_{b,\mathrm{os}}(X;{}^b\Omega^{\frac12})$ is thus an operator, which is $\mathbb{R}^+$-invariant, on the model space $\widetilde{X}$. As will be shown eventually, the Fredholm properties of P, given that it is elliptic, are captured by $I(P)$. Once again the map I is actually a homomorphism into the (non-commutative) algebra $\Psi^*_{b,I,\mathrm{os}}(\widetilde{X};{}^b\Omega^{\frac12})$. For the moment the full force of this result is not needed, however it is useful to observe:

LEMMA 4.45. *The passage to indicial operators gives a homomorphism*

$$I\colon\ \mathrm{Diff}^k_b(X;{}^b\Omega^{\frac12})\longrightarrow\!\!\!\!\rightarrow\mathrm{Diff}^k_{b,I}(\widetilde{X};{}^b\Omega^{\frac12}) \tag{4.100}$$

such that if $P\in\mathrm{Diff}^k_b(X;{}^b\Omega^{\frac12})$, *and* $A\in\Psi^m_{b,\mathrm{os}}(X;{}^b\Omega^{\frac12})$ *then*

$$I(PA)=I(P)\circ I(A). \tag{4.101}$$

PROOF: For the identity it is clear that $I(\mathrm{Id}) = \mathrm{Id}$. If x, y, x', y' are coordinates as usual and $s = x/x'$, $r = x + x'$ are projective coordinates near the front face of X_b^2 via the left action lifted to X_b^2

$$x\frac{\partial}{\partial x} \longmapsto s\frac{\partial}{\partial s} + r\frac{\partial}{\partial r},\ \frac{\partial}{\partial y_j} \longrightarrow \frac{\partial}{\partial y_j}.$$

The same is true for operators on $\widetilde{X}$. Clearly then if $A \in \Psi^m_{b,\mathrm{os}}(X; {}^b\Omega^{\frac12})$ the indicial operator satisfies

$$I(x\frac{\partial}{\partial x} \circ A) = s\frac{\partial}{\partial s}I(A),\ I(\frac{\partial}{\partial y_j} \circ A) = \frac{\partial}{\partial y_j}I(A).$$

By specializing A to the identity it follows that

$$\begin{aligned} P = \sum_{j+|\alpha|\le k} p_{j,\alpha}(x,y)(xD_x)^j D_y^\alpha \Longrightarrow \\ I(P) = \sum_{j+|\alpha|\le k} p_{j,\alpha}(0,y)(sD_s)^j D_y^\alpha, \end{aligned} \tag{4.102}$$

where the coordinates induced by x, y in $\widetilde{X}$ are used. This gives (4.100) and (4.101) follows as well.

Consider for a moment where this map comes from 'algebraically'. By definition $\mathrm{Diff}_b^*(X)$ is generated by $\mathcal{C}^\infty(X)$ and $\mathcal{V}_b(X)$ acting on $\mathcal{C}^\infty$ functions on X. For the 'coefficients' there is a natural map defined by restriction and lifting:

$$\mathcal{C}^\infty(X) \longrightarrow\!\!\!\!\!\rightarrow \mathcal{C}_I^\infty(\widetilde{X}),\ \phi \longmapsto \phi_{\restriction \partial X} \longmapsto \pi^*(\phi_{\restriction \partial X}) \in \mathcal{C}^\infty(\widetilde{X}), \tag{4.103}$$

where the subscript 'I' refers to invariance under the $\mathbb{R}^+$-action on $\widetilde{X}$.

There is a similar map for $\mathcal{V}_b(X)$. Since all elements of $\mathcal{V}_b(X)$ are tangent to the boundary the subspace

$$x\mathcal{V}_b(X) = \{V \in \mathcal{V}_b(X); V = xW,\ W \in \mathcal{V}_b(X)\}$$

is an ideal:

$$[\mathcal{V}_b(X), x\mathcal{V}_b(X)] \subset x\mathcal{V}_b(X),\ [V, xW] = xfW + x[V,W],\ f = \frac{Vx}{x}.$$

It follows that the quotient is also a Lie algebra

$$\mathcal{C}^\infty(\partial X; {}^bTX) = \mathcal{V}_b(X)/x\mathcal{V}_b(X).$$

In local coordinates this process is just freezing the coefficients at the boundary. The resulting map is a Lie algebra homomorphism

$$\mathcal{V}_b(X) \xrightarrow{I} \mathcal{C}^\infty(\partial X; {}^bTX). \tag{4.104}$$

The range space here can be reinterpreted:

LEMMA 4.46. *There is a natural isomorphism:*

$$\mathcal{C}^\infty(\partial X; {}^bTX) \cong \mathcal{V}_{b,I}(\widetilde{X}), \tag{4.105}$$

where $\widetilde{X}$ is the compactified normal bundle to ∂X and $\mathcal{V}_{b,I}(\widetilde{X}) \subset \mathcal{V}_b(\widetilde{X})$ is the subspace consisting of the $\mathbb{R}^+$-invariant elements for the action (4.94).

PROOF: Use of a partition of unity on ∂X reduces the question to a local one, with coordinates x, y. Then both sides of (4.105) are locally spanned over $\mathcal{C}^\infty(\partial X)$ by the basis vector fields $x\partial/\partial x$ and $\partial/\partial y_j$. Moreover this isomorphism is clearly independent of the choice of coordinates.

One can give a more invariant proof by noting that both sides of (4.105) have subalgebras which are commutative and isomorphic to $\mathcal{C}^\infty(\partial X)$. On the left it is just the span of $x\partial/\partial x$, which is a well-defined section of ${}^bT_{\partial X}X$, and on the right it is the subspace tangent to the fibres of $\widetilde{X}$ as a $[-1,1]$-bundle over ∂X. Both quotients are just $\mathcal{C}^\infty(\partial X; T\partial X)$ and the actions giving decompositions as semidirect products are also isomorphic.

Thus (4.104) can be written

$$I : \mathcal{V}_b(X) \longrightarrow\!\!\!\!\!\rightarrow \mathcal{V}_{b,I}(\widetilde{X}) \tag{4.106}$$

giving a Lie algebra homomorphism consistent with (4.103). The extension of (4.106) to the enveloping algebras gives the indicial operator in the differential case:

$$I : \mathrm{Diff}^k_b(X) \longrightarrow\!\!\!\!\!\rightarrow \mathrm{Diff}^k_{b,I}(\widetilde{X}) \ \forall\ k. \tag{4.107}$$

Clearly (4.102) just gives the local coordinate form of this map.

EXERCISE 4.47. Make sure that you can visualize these operations on X^2_b and that you can see why (4.107) is the same map as (4.100) except for the density factor.

4.16. General coefficients.

The calculus of b-pseudodifferential operators has been developed purely for operators acting on b-half-densities. This restriction is completely arbitrary and was made up to this point simply to limit the notational overhead. The extension to the case of operators on sections of general vector bundles involves nothing essentially new and is described here as much to give a summary of the results proved so far for the small calculus (and a few not proved until later) as from real necessity.

Let E and F be vector bundles over a compact manifold with boundary. Rather than repeat the definition by localization, one can take as the space of kernels the space of finite linear combinations of products of kernels in the b-half-density case and smooth homomorphisms. If $\operatorname{Hom}(E,F)$ is the bundle over X^2 with fibres $\hom(E_z,F_{z'})$ at the point $(z,z')\in X^2$ then

$$\begin{gathered}\Psi^m_{b,\mathrm{os}}(X;E,F)= \\ \Psi^m_{b,\mathrm{os}}(X;{}^b\Omega^{\frac12})\otimes_{\mathcal{C}^\infty(X^2_b)}\mathcal{C}^\infty(X^2_b;\beta_b^*\operatorname{Hom}(F\otimes{}^b\Omega^{-\frac12},E\otimes{}^b\Omega^{-\frac12})).\end{gathered}\tag{4.108}$$

For the case $E=F={}^b\Omega^{\frac12}$ the second factor here is just $\mathcal{C}^\infty(X^2_b)$, so the definition reduces to the original one in that case. By localization to neighbourhoods where the bundles are trivial it is easy to extend the results above. Thus:

$$\begin{aligned}A\in\Psi^m_{b,\mathrm{os}}(X;E,F)\Longrightarrow& \\ &A\colon\dot{\mathcal{C}}^\infty(X;E)\longrightarrow\dot{\mathcal{C}}^\infty(X;F)\\ &A\colon\mathcal{C}^\infty(X;E)\longrightarrow\mathcal{C}^\infty(X;F)\\ &A\colon\mathcal{C}^{-\infty}(X;E)\longrightarrow\mathcal{C}^{-\infty}(X;F).\end{aligned}$$

Composition of such operators makes sense, and the results in §5.9 extend to give

$$\Psi^m_{b,\mathrm{os}}(X;G,F)\circ\Psi^{m'}_{b,\mathrm{os}}(X;E,G)\subset\Psi^{m+m'}_{b,\mathrm{os}}(X;E,F).$$

The symbol map gives a short exact sequence

$$\begin{aligned}0\longrightarrow\Psi^{m-1}_{b,\mathrm{os}}(X;E,F)\hookrightarrow&\Psi^m_{b,\mathrm{os}}(X;E,F)\xrightarrow{{}^b\sigma_m}\\ &S^{[m]}({}^bT^*X;\pi^*\hom(E,F))\longrightarrow 0,\end{aligned}\tag{4.109}$$

where $\hom(E,F)$ is the bundle over X with fibres $\hom(E_z,F_z)$ at $z\in X$ and $\pi\colon{}^bT^*X\longrightarrow X$ is the projection. Furthermore the symbol map is multiplicative

$$\begin{gathered}{}^b\sigma_{m+m'}(A\circ B)={}^b\sigma_m(A)\cdot{}^b\sigma_{m'}(B),\\ A\in\Psi^m_{b,\mathrm{os}}(X;G,F),\ B\in\Psi^{m'}_{b,\mathrm{os}}(X;E;G).\end{gathered}\tag{4.110}$$

It will be shown below that the indicial homomorphism extends to

$$I\colon\Psi^m_{b,\mathrm{os}}(X;E,F)\longrightarrow\Psi^m_{b,I,\mathrm{os}}(\widetilde{X};E,F),$$

where in the image E and F are really the restrictions of E and F to the boundary of X pulled back to the compactified normal bundle $\widetilde{X}$ by the natural projection from $\widetilde{X}$ to ∂X. The corresponding short exact sequence is

$$0 \longrightarrow \rho\Psi^m_{b,\mathrm{os}}(X;E,F) \hookrightarrow \Psi^m_{b,\mathrm{os}}(X;E,F) \\ \longrightarrow \Psi^m_{b,I,\mathrm{os}}(\widetilde{X};E,F) \longrightarrow 0.$$

This is also multiplicative

$$I(A \circ B) = I(A) \circ I(B), \tag{4.111}$$

as in (4.110). There is a connection between the symbol and indicial homomorphisms:

$$^b\sigma_m(A)_{\restriction {}^bT^*_{\partial X}X} = {}^b\sigma_m(I(A))_{\restriction {}^bT^*_{\partial_0 X}\widetilde{X}}$$

where $\partial_0 X$ is one of the boundary faces of $\widetilde{X}$; this uses the the natural identification of the b-cotangent bundles of X and $\widetilde{X}$ over their boundaries.

EXERCISE 4.48. Show that the definition of the indicial homomorphism for differential operators, in (4.102), does indeed extend to a well-defined map

$$I\colon \operatorname{Diff}^k_b(X;E,F) \longrightarrow \operatorname{Diff}^k_{b,I}(\widetilde{X};E,F)$$

for any vector bundles E, F. Check that it has the composition property (4.111). If E and F are Hermitian and X is equipped with a non-vanishing b-density, ν, then the adjoint of an operator is defined by

$$\int_X \langle P^* f, e\rangle \nu = \int_X \langle f, Pe\rangle \ \forall\ e \in \dot{\mathcal{C}}^\infty(X;E),\ f \in \dot{\mathcal{C}}^\infty(X;F).$$

Show that the boundary value of ν fixes a non-vanishing, $\mathbb{R}^+$-invariant, b-density on $\widetilde{X}$ with respect to which the adjoint of a differential operator satisfies

$$I(P^*) = [I(P)]^*. \tag{4.112}$$

4.17. Examples.

Having defined the indicial operator in general, at least for b-differential operators, it is opportune to compute it for the examples considered so far. Choosing a trivialization of $N\partial X$, i.e. the differential, dx, at ∂X of a boundary defining function, reduces $\widetilde{X}$ to $[-1,1] \times \partial X$. Reverting to the non-compact variable x on $N\partial X$, a general element $Q \in \operatorname{Diff}^m_{b,I}(\widetilde{X};E,F)$ is of the form

$$Q = \sum_{k \le m} Q_k (x\frac{\partial}{\partial x})^k,\ Q_k \in \operatorname{Diff}^{m-k}(\partial X; E_{\restriction \partial X}, F_{\restriction \partial X}).$$

Consider first the exterior differential in (2.21). The choice of trivialization of the normal bundle fixes the section dx/x of ${}^bT^*_{\partial X}X$ which is complementary to $T^*\partial X \subset {}^bT^*_{\partial X}X$ and so gives a decomposition

$$ {}^bT^*_{\partial X}X = \operatorname{span}(\frac{dx}{x}) \oplus T^*\partial X. $$

Taking exterior powers

$$ {}^b\Lambda^k_{\partial X}X \equiv \Lambda^k\partial X \oplus \Lambda^{k-1}\partial X,\ u = u' + \frac{dx}{x} \wedge u''. \tag{4.113} $$

Using this the indicial operator of d becomes a 2×2 matrix (except for $k = 0$ and $k = \dim X$)

$$ I(d) = \begin{pmatrix} d & 0 \\ x\frac{\partial}{\partial x} & -d \end{pmatrix}, $$

where on the right d is the exterior differential on the boundary.

Applying Exercise 4.48 to the adjoint, δ, of d with respect to an exact b-metric it follows easily that

$$ I(\delta) = \begin{pmatrix} \delta & -x\frac{\partial}{\partial x} \\ 0 & -\delta \end{pmatrix}, $$

provided the decomposition (4.113) is taken with respect to the form dx/x defined by the metric. Here δ on the right is the adjoint of δ with respect to the metric induced on the boundary.

For the Laplacian of an exact b-metric in b-forms it follows that

$$ I({}^b\Delta) = \begin{pmatrix} \Delta_0 + (xD_x)^2 & 0 \\ 0 & \Delta_0 + (xD_x)^2 \end{pmatrix}, $$

where Δ_0 is the Laplacian on the boundary.

Next consider the Levi-Civita connection of an exact b-metric, acting on any associated bundle. Here the trivialization should be chosen so that the metric takes the form (2.11) with respect to it. Then

$$ \begin{aligned} I(\nabla) = (x\frac{\partial}{\partial x}, \nabla_0) \colon \mathcal{C}^\infty(\widetilde{X}; E) &\longrightarrow \mathcal{C}^\infty(\widetilde{X}; {}^bT^*_{\partial X}X \otimes E) \\ &= \mathcal{C}^\infty(\widetilde{X}; E) \oplus \mathcal{C}^\infty(\widetilde{X}; T^*\partial X \otimes E), \end{aligned} \tag{4.114} $$

where ∇_0 is the induced connection on the boundary.

EXERCISE 4.49. Prove (4.114).

From (3.71) the indicial operator of the twisted Dirac operator on an even-dimensional exact b-spin manifold can be found. It is an operator

$$I(\eth^{\pm}_{E})\colon \mathcal{C}^\infty(\widetilde{X};{}^{\pm}S) \longrightarrow \mathcal{C}^\infty(\widetilde{X};{}^{\mp}S).$$

Recalling that over the boundary the spinor bundles are isomorphic to the spinor bundle of the boundary,

$$M_{\pm}\colon {}^{\pm}S \longleftrightarrow S_0 \text{ over } \partial X.$$

In terms of this

$$I(\eth^{\pm}_{E}) = M_{\mp}^{-1}\left(\pm x\frac{\partial}{\partial x} + \eth_0\right) M_{\pm} \tag{4.115}$$

which is just as in (In.23).

EXERCISE 4.50. Prove (4.115).

This formula can be extended to the generalized Dirac operators considered in Lemma 3.32 and the Corollary to Lemma 3.34. Consider the homomorphism (2.97) restricted to the boundary:

$$\nabla_{x\partial/\partial x} \in \hom(E_{\restriction\partial X}). \tag{4.116}$$

By assumption this preserves the grading. Moreover since the connection is assumed to be Clifford, it commutes with Clifford multiplication by $\frac{dx}{x}$ and so induces an isomorphism on the induced Clifford module on the boundary; denote this by τ. Since the extra term in the indicial operator, as opposed to the boundary operator computed in (3.72), arises from $-i\,\mathrm{cl}(\frac{dx}{x})\nabla_{x\partial/\partial x}$ it follows that:

LEMMA 4.51. *For the generalized Dirac operators of Lemma 3.32 the indicial operator is*

$$I({}^b\eth_E) = \begin{pmatrix} 0 & I({}^b\eth_{-,E}) \\ I({}^b\eth_{+,E}) & 0 \end{pmatrix},$$

where in place of (4.115)

$$I({}^b\eth_{\pm,E}) = M_{\mp}^{-1}\left(\pm x\frac{\partial}{\partial x} + \tau + \eth_0\right) M_{\pm}.$$

4.18. Trace class operators.

Although it is clear from the discussion in §4.14 that a b-pseudodifferential operator of order $-\infty$ is compact only if its indicial operator vanishes, it is very useful to extend the trace functional to these operators. Recall that for $N \times N$ matrices the trace is the sum of the diagonal entries

$$\operatorname{tr}(A) = \sum_{i=1}^{N} a_{ii}, \quad A = (a_{ij})_{i,j=1}^{N}. \tag{4.117}$$

If A and B are $N \times N$ matrices then the commutator is traceless:

$$\begin{gathered} \operatorname{tr}([A, B]) = 0,\ [A, B] = AB - BA \\ [A, B]_{ij} = \sum_{k=1}^{N} (a_{ik} b_{kj} - b_{ik} a_{kj}), \end{gathered} \tag{4.118}$$

as follows directly from the definition. Thus if T is an invertible matrix then

$$\operatorname{tr}(TAT^{-1}) = \operatorname{tr}(A) \quad \forall\ A, \tag{4.119}$$

since $TAT^{-1} - A = [TA, T^{-1}]$. The trace is therefore well-defined as a linear functional on the space of (continuous) linear operators on any finite dimensional vector space. The Jordan normal form shows that if the eigenvalues of A are summed with their multiplicities

$$m(\lambda) = \lim_{k \to \infty} \dim \operatorname{null}(A - \lambda)^k, \quad \lambda \in \mathbb{C}$$

then

$$\operatorname{tr}(A) = \sum_{\lambda \in \mathbb{C}} m(\lambda)\lambda, \tag{4.120}$$

where the sum is finite for a linear operator on a finite dimensional space. On a Hilbert space, H, the finite rank operators are those with kernels of the form

$$A = \sum_{j=1}^{N} e_j f_j, \quad e_j, f_j \in H. \tag{4.121}$$

Conventionally they act through

$$A\phi = \sum_{j=1}^{N} \langle \phi, f_j \rangle e_j,$$

and hence act on the finite dimensional subspace spanned by the e_j and f_j. The trace, given by (4.120) on this subspace, is then

$$\operatorname{tr}(A) = \sum_{j=1}^{N} \langle e_j, f_j \rangle. \tag{4.122}$$

EXERCISE 4.52. Check that (4.122) holds and is independent of choices.

The trace norm of A, of finite rank, is

$$\|A\|_{\mathrm{Tr}} = \sup \sum_{j=1}^{N} |\langle Af_j, e_j\rangle|, \tag{4.123}$$

with the supremum taken over all pairs of orthonormal bases, $\{e_j\}$ and $\{f_j\}$ of H. Again (4.118) and (4.119) hold for finite rank operators. If H is a Hilbert space the space of trace class operators on H is the closure of the finite rank operators with respect to the norm (4.123).

EXERCISE 4.53. Check that (4.123) does indeed define a norm. Show that, for finite rank operators,

$$|\operatorname{tr}(A)| \le \|A\|_{\mathrm{Tr}}$$

and hence conclude that the functional tr extends by continuity in the trace norm to all trace class operators.

If A has finite rank and B is a bounded operator on H then both AB and BA have finite rank and, from (4.118), $\operatorname{tr}[A, B] = 0$. It can be shown that

$$\|AB\|_{\mathrm{Tr}} \le \|A\|_{\mathrm{Tr}}\|B\|$$

with $\|B\|$ the norm of B as a bounded operator. This and a simple convergence argument lead to the following result:

PROPOSITION 4.54. *The trace class operators on a Hilbert space form an ideal in the bounded operators with* $\operatorname{tr}[A, B] = 0$ *if A is trace class and B bounded and hence* $\operatorname{tr}(TAT^{-1}) = \operatorname{tr}(A)$ *if A is trace class and T is invertible.*

The relevant Hilbert spaces here are Sobolev spaces, in particular spaces of square-integrable functions. For compact manifolds without boundary the following form of Lidskii's theorem gives an alternative way of computing the trace.

PROPOSITION 4.55. *On a compact manifold without boundary, X, the smoothing operators are trace class and*

$$\operatorname{tr}(A) = \int_X K_A(x, x), \quad A \in \Psi^{-\infty}(X; \Omega^{\frac{1}{2}}). \tag{4.124}$$

By definition the kernel, K_A, is a $\mathcal{C}^\infty$ half-density on X^2. Restricted to $\Delta \subset X^2$, the diagonal, the half-density bundle of X^2 is canonically isomorphic to the density bundle on X, so (4.124) makes sense. Thus

$$
\begin{aligned}
\Omega^{\frac{1}{2}}(X^2)_{\restriction\Delta} &\cong \Omega^{\frac{1}{2}}(\Delta)\otimes\Omega^{\frac{1}{2}}_{\text{fibre}}(N\Delta)\\
&\cong \Omega^{\frac{1}{2}}(X)\otimes\Omega^{\frac{1}{2}}_{\text{fibre}}(TX)\\
&\cong \Omega^{\frac{1}{2}}(X)\otimes\Omega^{\frac{1}{2}}(X)\\
&\cong \Omega(X).
\end{aligned}
\tag{4.125}
$$

Here the isomorphisms for densities discussed in Chapter 2 are used successively.

EXERCISE 4.56. Prove (4.124) following Hörmander [46,47] (or otherwise). Starting again with finite rank operators, on $L^2(X;\Omega^{\frac{1}{2}})$, consider the Hilbert-Schmidt norm:

$$
||A||_{\text{HS}} = \Big(\sum_{i,j}|a_{ij}|^2\Big)^{\frac{1}{2}} = \Big(\sum_j \int_X |e_j|^2\int_X |f_j|^2\Big)^{\frac{1}{2}}
\tag{4.126}
$$

in terms of (4.117), (4.121) respectively where, in the latter case it can also be assumed that $e_1,\dots e_N$ and $f_1,\dots f_N$ are (separately) orthogonal families. Relate this to (4.123) by showing that for finite rank operators

$$
||A||_{\text{Tr}} \le ||B||_{\text{HS}}||C||_{\text{HS}},\quad A = BC.
$$

Show that defining the Hilbert-Schmidt operators as the closure of the finite rank operators with respect to (4.126) gives an ideal which, for $H = L^2(X;\Omega^{\frac{1}{2}})$, is exactly the space of operators with square integrable *kernels* and

$$
||A||^2_{\text{HS}} = \int_{X^2}|K(x,y)|^2.
$$

Show that if $A = BC$, with B and C Hilbert-Schmidt, then A is trace class, and

$$
\operatorname{tr}(A) = \int_{X^2} K_B(x,y)K_C(y,x).
\tag{4.127}
$$

Check that any $B\in\Psi^{-\frac{n}{2}-\epsilon}(X;\Omega^{\frac{1}{2}}), \epsilon>0, n=\dim X$, is Hilbert-Schmidt. From the composition formula for pseudodifferential operators (or otherwise) show that *any* $A\in\Psi^{-n-\epsilon}(X;\Omega^{\frac{1}{2}})$ is trace class and deduce (4.124) from (4.127).

This is a lot of work, but it is important to get to the case of most interest here, that of a compact manifold with boundary. As already observed the direct generalization of Proposition 4.55 to the b-calculus does *not* hold; namely the operators of order $-\infty$ are *not* in general trace class; they are not even compact in general.

PROPOSITION 4.57. *If X is a compact manifold with boundary then an operator $A \in \Psi_b^{-\infty}(X; {}^b\Omega^{\frac12})$ is trace class on $L^2(X; {}^b\Omega^{\frac12})$ if and only if*

$$I(A) = 0 \tag{4.128}$$

and then

$$\operatorname{Tr}(A) = \int_X (K_A)_{\restriction \Delta_b}. \tag{4.129}$$

Similar remarks apply here as in Lidskii's Theorem, Proposition 4.55, but with an important twist. The kernel, written $K_A \in \mathcal{C}^\infty(X_b^2; {}^b\Omega^{\frac12})$, can be restricted to the lifted diagonal as a $\mathcal{C}^\infty$ section of ${}^b\Omega^{\frac12}(X_b^2)$. Recall that there is an isomorphism similar to that in (4.125). First, since Δ_b is transversal to the boundary (by Lemma 4.3),

$${}^b\Omega^{\frac12}(X_b^2)_{\restriction \Delta_b} \cong {}^b\Omega^{\frac12}(X) \otimes \Omega^{\frac12}_{\text{fibre}}(N\Delta_b), \tag{4.130}$$

where the fact that $\Delta_b \cong X$ has been used. In Lemma 4.6 the normal bundle to Δ_b is identified with ${}^bTX$ and clearly

$$\Omega^{\frac12}_{\text{fibre}}({}^bTX) \cong {}^b\Omega^{\frac12}(X),$$

so from (4.130)

$${}^b\Omega^{\frac12}(X_b^2)_{\restriction \Delta_b} \cong {}^b\Omega(X).$$

Thus the restriction of the kernel to Δ_b defines a section

$$(K_A)_{\restriction \Delta_b} \in \mathcal{C}^\infty(X; {}^b\Omega). \tag{4.131}$$

As discussed in §4.19 a general smooth section of ${}^b\Omega$ cannot be integrated over X. However the condition (4.128) means that in (4.131) the section vanishes at the boundary. So, given (4.128), $(K_A)_{\restriction \Delta_b} \in \mathcal{C}^\infty(X; \Omega)$ and (4.129) is meaningful.

PROOF: The most important part is the sufficiency, that (4.128) implies that $A \in \Psi_b^{-\infty}(X; {}^b\Omega^{\frac12})$ is trace class. From the boundedness properties of b-pseudodifferential operators, discussed in the next chapter, it follows that

$$A\colon L^2(X;{}^b\Omega^{\frac12}) \longrightarrow H_b^\infty(X;{}^b\Omega^{\frac12})\ \forall\ A \in \Psi_b^{-\infty}(X;{}^b\Omega^{\frac12}). \tag{4.132}$$

The vanishing condition (4.128) means precisely that $A = xB$, where B is an element of $\Psi_b^{-\infty}(X;{}^b\Omega^{\frac12})$. To prove that A is then trace class, it is enough to use the obvious extension of (4.124) to manifolds with boundary. Namely if p is large enough then

$$K \in \dot{\mathcal{C}}^p(X^2;\Omega^{\frac12}) \Longrightarrow K \text{ is trace class and } \operatorname{tr}(K) = \int\limits_X K_{\restriction\Delta}. \tag{4.133}$$

Of course the problem is that the kernel of A is certainly *not* this smooth, just because of (4.128). However there is a trick to make it so.

Given a positive integer p and a boundary defining function $\rho \in \mathcal{C}^\infty(X)$ for ∂X, set

$$\begin{aligned}\mathcal{C}^\infty(X_{p^{-1}}) = \big\{u\colon X \longrightarrow \mathbb{C};\, u = F(u_1,\dots,u_q,\rho^{\frac1p}),\\ u_1,\dots,u_q \in \mathcal{C}^\infty(X), F \in \mathcal{C}^\infty(\mathbb{R}^{q+1})\big\}.\end{aligned} \tag{4.134}$$

It is straightforward to check that this space of functions turns X into a $\mathcal{C}^\infty$ manifold, denoted $X_{p^{-1}}$. As a point-set $X_{p^{-1}}$ and X are identical. Differentially however there are 'more' $\mathcal{C}^\infty$ functions on $X_{p^{-1}}$; it is in fact X with the boundary blown up to pth order. Directly from (4.134) the trivial map

$$\beta_p\colon X_{p^{-1}} \longrightarrow X \text{ is } \mathcal{C}^\infty. \tag{4.135}$$

EXERCISE 4.58. Show that, if $p > 1$, (4.135) is *not* an isomorphism. However, show that X and $X_{p^{-1}}$ *are* always isomorphic (but there is no natural isomorphism).

Although (4.135) is not an isomorphism it does have many properties of an isomorphism:

LEMMA 4.59. *Under β_p each non-vanishing section of ${}^b\Omega(X)$ lifts to a non-vanishing $\mathcal{C}^\infty$ section of ${}^b\Omega(X_{p^{-1}})$, also $\mathcal{V}_b(X)$ lifts into $\mathcal{V}_b(X_{p^{-1}})$ and β_p induces an isomorphism*

$$\begin{aligned}\beta_p^*&\colon L^2(X;{}^b\Omega^{\frac12}) \longleftrightarrow L^2(X_{p^{-1}};{}^b\Omega^{\frac12})\\ \beta_p^*&\colon H_b^\infty(X;{}^b\Omega^{\frac12}) \longleftrightarrow H_b^\infty(X_{p^{-1}};{}^b\Omega^{\frac12}).\end{aligned} \tag{4.136}$$

PROOF: If $\rho \in \mathcal{C}^\infty(X)$ is a defining function for ∂X then $\beta_p^*\rho = x^p$, with $x \in \mathcal{C}^\infty(X_{p^{-1}})$ a defining function for $\partial X_{p^{-1}}$. Thus $\beta_p^*(d\rho/\rho) = pdx/x$, from which the statement about sections of ${}^b\Omega$ follows. If $\rho, y_1, \dots, y_n$ are local coordinates near ∂X then $x, y_1, \dots, y_n$ are local coordinates in the preimage, near $\partial X_{p^{-1}}$. Then

$$(\beta_p)_*(x\partial_x) = p\rho\partial_\rho,\ (\beta_p)_*(\partial_{y_j}) = \partial_{y_j}.$$

This shows that $\mathcal{V}_b(X)$ lifts into (and spans) $\mathcal{V}_b(X_{p^{-1}})$. Then (4.136) is immediate.

From (4.136) and the invariance of the trace class, trace class operators are the same on $L^2(X; {}^b\Omega^{\frac{1}{2}})$ and $L^2(X_{p^{-1}}; {}^b\Omega^{\frac{1}{2}})$. From (4.132), if $A \in x\Psi^{-\infty}(X; {}^b\Omega^{\frac{1}{2}})$ then

$$\begin{aligned} A\colon L^2(X; {}^b\Omega^{\frac{1}{2}}) &\longrightarrow \rho H_b^\infty(X; {}^b\Omega^{\frac{1}{2}}) \Longrightarrow \\ A\colon L^2(X_{p^{-1}}; {}^b\Omega^{\frac{1}{2}}) &\longrightarrow x^p H_b^\infty(X_{p^{-1}}; {}^b\Omega^{\frac{1}{2}}). \end{aligned}$$

In fact it is easy to see that under β_p

$$\Psi_b^{-\infty}(X; {}^b\Omega^{\frac{1}{2}}) \longrightarrow \Psi_b^{-\infty}(X_{p^{-1}}; {}^b\Omega^{\frac{1}{2}}). \tag{4.137}$$

EXERCISE 4.60. Prove (4.137) by showing that the blow-down map β_p, acting on each factor, extends from the interior of $(X_{p^{-1}})^2_b$ to define a $\mathcal{C}^\infty$ map $\beta_p^2\colon (X_{p^{-1}})^2_b \longrightarrow X^2_b$.

Taking p sufficiently large, (4.133) applies to $A \in \rho\Psi_b^{-\infty}(X; {}^b\Omega^{\frac{1}{2}})$ as an operator on $X_{p^{-1}}$ so A is trace class and the sufficiency of (4.128) for (4.129) follows. The necessity in Proposition 4.57 will be discussed below.

There are more elementary ways to prove the sufficiency of (4.128). However this blow-up argument seems to be appropriate technology and also β_p is an interesting example of a map which in the b-category is close to an isomorphism.

EXERCISE 4.61. Show that β_p is a b-fibration in the sense of Exercise 4.7 and so is the map β_p^2 of Exercise 4.60.

4.19. The b-integral.

Having observed that general elements of $\Psi_b^{-\infty}(X; {}^b\Omega^{\frac{1}{2}})$ are *not* trace class it is nevertheless important for present purposes to define their traces! To do this the first step is to define an extension of the integral, a b-integral defined on $\mathcal{C}^\infty(X; {}^b\Omega)$. This is a regularization, in the sense of Hadamard, of the generally divergent integral. Recall that restriction to the boundary is a natural map

$$\begin{gathered}\mathcal{C}^\infty(X;{}^b\Omega)\ni\phi\longmapsto\phi_{\restriction\partial X}\in\mathcal{C}^\infty(\partial X;\Omega)\\ \psi\cdot\left|\frac{dx}{x}dy\right|\longmapsto\psi_{\restriction\partial X}|dy|.\end{gathered}$$

LEMMA 4.62. *Let $\nu\in\mathcal{C}^\infty(\partial X;{}_+N\partial X)$ be a trivialization of the normal bundle of a compact manifold with boundary and suppose $x\in\mathcal{C}^\infty(X)$ is a defining function for ∂X with $dx\cdot\nu=1$ at ∂X, then*

$$\tag{4.138} {}^{\nu}\!\!\int_X\phi=\lim_{\epsilon\downarrow 0}\left[\int_{x>\epsilon}\phi+\log\epsilon\cdot\int_{\partial X}\phi_{\restriction\partial X}\right]$$

exists for all $\phi\in\mathcal{C}^\infty(X;{}^b\Omega)$ and depends only on ϕ and ν. If $\nu'=a\nu$, $0<a\in\mathcal{C}^\infty(\partial X)$, is another trivialization then

$$\tag{4.139} {}^{\nu'}\!\!\int_X\phi-{}^{\nu}\!\!\int_X\phi=\int_{\partial X}\log a\cdot(\phi_{\restriction\partial X}).$$

PROOF: Choose $\delta>0$ small so that

$$\{x\le\delta\}\cong\partial X\times[0,\delta]_x.$$

Then, in $x\le\delta$, $\phi=\psi\cdot\frac{dx}{x}\mu$, $\mu\in\mathcal{C}^\infty(\partial X;\Omega)$, $\psi\in\mathcal{C}^\infty(X)$, so

$$\begin{aligned}\int_{x>\epsilon}\phi&=\int_{x\ge\delta}\phi+\int_{\partial X}\int_{x=\epsilon}^{x=\delta}\psi(x,\cdot)\frac{dx}{x}\mu\\ &=\int_{x\ge\delta}\phi+\int_{\partial X}\left(\int_\epsilon^\delta\left[\frac{\psi(x,\cdot)-\psi(0,\cdot)}{x}\right]dx\right)\mu+(\log\frac{\delta}{\epsilon})\int_{\partial X}\phi_{\restriction\partial X}\end{aligned}$$

since $\psi(0,\cdot)\mu=\phi_{\restriction\partial X}$. Clearly the terms on the right, except the last, converge as $\epsilon\downarrow 0$, so the limit in (4.138) does exist.

If $x'=a(x,y)x$ is another defining function then

$$\int_{x'>\epsilon}\phi-\int_{x>\epsilon}\phi=\int_{\partial X}\int_{x=\epsilon}^{xa(x,y)=\epsilon}\psi\frac{dx}{x}\mu\longrightarrow\int_{\partial X}\phi_{\restriction\partial X}\log a(0,y)\text{ as }\epsilon\downarrow 0.$$

This proves (4.139).

EXERCISE 4.63. In §5.1 the indicial family of a b-pseudodifferential operator is introduced. For a b-differential operator it can be defined in terms of the representation (4.102) by replacing xD_x by the variable λ :

$$I_\nu(P,\lambda) = \sum_{j+|\alpha|\le k} p_{j,\alpha}(0,y)\lambda^j D_y^\alpha,$$

where the suffix ν refers to the choice of trivialization of the normal bundle which is involved. Show that with this definition

$$\begin{gathered} {}^{\nu}\!\!\int Pu\cdot\overline{v} - {}^{\nu}\!\!\int u\cdot\overline{P^*v} = \frac{1}{i}\int_{\partial X}\partial_\lambda I_\nu(P,0)u\cdot\overline{v} \\ \forall\ u,v\in\mathcal{C}^\infty(X;{}^b\Omega^{\frac12}),\ P\in\mathrm{Diff}_b^k(X;{}^b\Omega^{\frac12}). \end{gathered} \tag{4.140}$$

4.20. The b-trace functional.

Having defined the regularized integral the b-trace can now be defined:

DEFINITION 4.64. If $\nu\in\mathcal{C}^\infty(\partial X;{}_+N\partial X)$ is a trivialization of the normal bundle then for $A\in\Psi_b^{-\infty}(X;{}^b\Omega^{\frac12})$

$$\text{b-Tr}_\nu(A) = {}^{\nu}\!\!\int_X A_{\restriction\Delta_b}.$$

Now, from (4.138), observe that

$${}^{\nu}\!\!\int_X\phi = \int_X\phi\quad\ \forall\ \phi\in\mathcal{C}^\infty(X;{}^b\Omega),\ \phi_{\restriction\partial X}=0.$$

Thus from Proposition 4.57 it follows that

$$A\in\Psi_b^{-\infty}(X;{}^b\Omega^{\frac12}))\text{ trace class }\Longrightarrow\text{b-Tr}_\nu(A)=\mathrm{Tr}(A)\ \forall\ \nu.$$

This means that these b-trace functionals

$$\text{b-Tr}_\nu\colon\Psi_b^{-\infty}(X;{}^b\Omega^{\frac12})\longrightarrow\mathbb{C} \tag{4.141}$$

are *extensions* of the usual trace functional in the following sense. In the short exact sequence (4.99) for $m=-\infty$ the trace functional is defined on the first space; the b-trace functional is defined on the middle space and vanishes identically on the space of $\mathbb{R}^+$-invariant operators in which the indicial operator takes values. Thus the diagramme:

$$\begin{tikzcd} \rho_{\mathrm{bf}}\Psi^m_{b,\mathrm{os}}(X;{}^b\Omega^{\frac12}) \arrow[r]\arrow[d,"\mathrm{Tr}"] & \Psi^m_{b,\mathrm{os}}(X;{}^b\Omega^{\frac12}) \arrow[r,"I"]\arrow[d,"\text{b-Tr}_\nu"] & \Psi^m_{b,I,\mathrm{os}}(\widetilde{X};{}^b\Omega^{\frac12}) \arrow[d] \\ \mathbb{C} \arrow[r] & \mathbb{C} \arrow[r] & \{0\} \end{tikzcd}$$

commutes.

EXERCISE 4.65. Check this last statement that

$$\text{b-Tr}_\nu(A) = 0 \ \forall \ A \in \Psi^{-\infty}_{b,I,\mathrm{os}}(\widetilde{X}; {}^b\Omega^{\frac{1}{2}}).$$

The trace of a commutator, for trace class operators, is always zero, so it is reasonable to expect the trace of a commutator to be expressible in terms of the indicial operators. The fundamental identity established in §5.5 shows that the b-trace does not vanish on the commutator subspace and so implies that is not *really* a trace functional.

Chapter 5. Full calculus

In Chapter 4 the small b-pseudodifferential calculus was used to carry out the ('symbolic') construction of a parametrix. This gives an inverse up to errors in the small-residual space, $\Psi_b^{-\infty}(X; {}^b\Omega^{\frac{1}{2}})$. The fact that these operators are *not* generally compact shows that more needs to be done in order to understand the mapping, especially Fredholm, properties of elliptic b-differential operators. What is required is the inversion of the other symbol, i.e. the indicial operator. The Mellin transform can be applied, so reducing the question to the invertibility properties of the indicial family. For this the symbolic construction is a convenient starting point.

The structure of the inverse of the indicial operator is used as a guide to the extension of the calculus so that it includes a parametrix, up to compact errors. The composition properties of the extended calculus are then analyzed and the structure of generalized inverses is discussed.

The extensive discussion of the polyhomogeneous calculus in the latter part of this chapter leads to a quite detailed description of the kernels of such generalized inverses. Rather little of this is needed for a minimal proof of the APS theorem. In particular nothing from §5.18 onward is required, except for the extension, as in §5.24, of the calculus with bounds to general bundles.

5.1. Mellin transform.

To examine the invertibility properties of $I(P)$, the indicial operator of an elliptic b-differential operator as in (4.102), or just an elliptic element $Q \in \mathrm{Diff}^k_{b,I}(\widetilde{X})$, it is natural to use the *Mellin transform*

$$u_{M,\nu}(\lambda, y) = \int_0^\infty x^{-i\lambda} u(x,y) \frac{dx}{x},\ \ u \in \dot{\mathcal{C}}^\infty(\widetilde{X}). \tag{5.1}$$

Here $\nu : \widetilde{X} \longleftrightarrow [-1,1] \times \partial X$ is a (projective) trivialization of $\widetilde{X}$ and x is the corresponding linear coordinate. Thus some defining function $\rho \in \mathcal{C}^\infty(X)$ for ∂X fixes ν and x is just the linear function $d\rho$ on the interior of $\widetilde{X}$. Since

$$u \in \dot{\mathcal{C}}^\infty(\widetilde{X}) \Longleftrightarrow \sup_{\widetilde{X}} |x^p D_x^q T u| < \infty \quad \forall\ p \in \mathbb{Z}, q \in \mathbb{N}_0, T \in \mathrm{Diff}^*(\partial X),$$

the integral in (5.1) certainly converges absolutely. Indeed the Paley-Wiener theorem characterizes the range:

THEOREM 5.1. *The Mellin transform, (5.1), is an isomorphism*

$$
\begin{aligned}
\dot{\mathcal{C}}^\infty(\widetilde{X}) \ni u \longleftrightarrow u_{M,\nu} \in \Big\{ & U : \mathbb{C} \longrightarrow \mathcal{C}^\infty(\partial X) \text{ entire;} \\
& \sup_{|\operatorname{Im}\lambda| \le p} |(1+|\lambda|)^k \partial_\lambda^j TU| < \infty \ \forall\ k, j, p \in \mathbb{N}_0, T \in \operatorname{Diff}^*(\partial X) \Big\}
\end{aligned}
\tag{5.2}
$$

with inverse

$$
u(x,y) = \frac{1}{2\pi} \int_{\operatorname{Im}\lambda = r} x^{i\lambda} u_{M,\nu}(\lambda, y) d\lambda \tag{5.3}
$$

for any $r \in \mathbb{R}$.

EXERCISE 5.2. Review the proof of Theorem 5.1. First introduce $s = \log x$ as coordinate so that (5.1) becomes

$$
u_{M,\nu}(\lambda, y) = \int_{-\infty}^{\infty} e^{-is\lambda} u(e^s, y) ds. \tag{5.4}
$$

Show that $u \in \dot{\mathcal{C}}^\infty(\widetilde{X})$ is equivalent to the faster-than-any-exponential decay of all derivatives of $v(s,y) = u(e^s, y)$. Then recover (5.2) and (5.3) by reference to the standard results for the Fourier(-Laplace) transform.

Clearly the Mellin transform satisfies the identities

$$
\begin{aligned}
(xD_x u)_{M,\nu} &= \lambda u_{M,\nu} \text{ and} \\
(Tu)_{M,\nu} &= T u_{M,\nu},\ T \in \operatorname{Diff}^*(\partial X).
\end{aligned}
\tag{5.5}
$$

It follows that if $Q \in \operatorname{Diff}^k_{b,I}(\widetilde{X})$

$$
\begin{gathered}
Q = \sum_{j+|\alpha| \le k} q_{j,\alpha}(y) (xD_x)^j D_y^\alpha \Longrightarrow (Qu)_{M,\nu} = \widehat{Q} u_{M,\nu}, \\
\text{where } \widehat{Q}(\lambda) = \sum_{j+|\alpha| \le k} q_{j,\alpha}(y) \lambda^j D_y^\alpha.
\end{gathered}
\tag{5.6}
$$

If $P \in \operatorname{Diff}^k_b(X)$ is given by (4.102) then the operator

$$
I_\nu(P, \lambda) = \sum_{j+|\alpha| \le k} p_{j,\alpha}(0,y) \lambda^j D_y^\alpha = \widehat{I(P)} \tag{5.7}
$$

is the *indicial family* of P, already introduced in Exercise 4.63. It is a differential operator on ∂X depending parametrically, and polynomially, on $\lambda \in \mathbb{C}$.

5.2. Inversion of the indicial family.

Before proceeding to describe the basic result on the invertibility of the indicial family, note that $I_\nu(P,\lambda)$ depends a little on the choice of a trivialization, ν, of the normal bundle. Any other (oriented) trivialization, ν', corresponds to multiplication, on the fibres of $N\partial X$, by some $0 < a \in \mathcal{C}^\infty(\partial X)$, i.e. $x' = x/a$. Directly from (5.1) it follows that

$$I_{\nu'}(P,\lambda) = a^{i\lambda} I_\nu(P,\lambda) a^{-i\lambda}$$

is given by conjugation with the complex powers. Thus invertibility results will be independent of the choice of trivialization.

By the *order* of a pole of a meromorphic function, F, will be meant the negative of the most singular power which occurs in the Laurent series expansion around the singular point, i.e.

$$\operatorname{ord}(z) = \min\left\{k \in \mathbb{N}; (\lambda - z)^k F(\lambda) \text{ is holomorphic near } \lambda = z\right\}.$$

Thus

$$F(\lambda) = \sum_{j=1}^{\operatorname{ord}(z)} (\lambda - z)^{-j} A_j + F_0(\lambda) \tag{5.8}$$

with F_0 holomorphic near $\lambda = z$. The residue at z is A_1. On the other hand, for a meromorphic function $F(\lambda)$, with values in a space of linear operators, given as the inverse of a holomorphic family $Q(\lambda)$, the *rank* of a pole is the dimension of the singular range in the sense that

$$\begin{aligned} \operatorname{rank}(z) = \dim \Big\{ u = \sum_{j=1}^{\operatorname{ord}(z)} (\lambda - z)^{-j} u_j, \\ u_j \in \mathcal{C}^\infty(\partial X); Q(\lambda) u(\lambda, y) \text{ is holomorphic near } z \Big\}. \end{aligned} \tag{5.9}$$

Clearly $\operatorname{ord}(z) \le \operatorname{rank}(z)$. For a function with values in $\mathbb{C}$ the order and rank are always equal.

PROPOSITION 5.3. *If $Q \in \operatorname{Diff}^k_{b,I}(\widetilde{X})$ is elliptic then there is a discrete set*

$$\operatorname{spec}_b(Q) = \left\{\lambda \in \mathbb{C}; \widehat{Q}(\lambda) \text{ is not invertible on } \mathcal{C}^\infty(\partial X)\right\}$$

and the inverse is a meromorphic family

$$[\mathbb{C} \setminus \operatorname{spec}_b(Q)] \ni \lambda \longmapsto Q(\lambda)^{-1} \in \Psi^{-k}(\partial X)$$

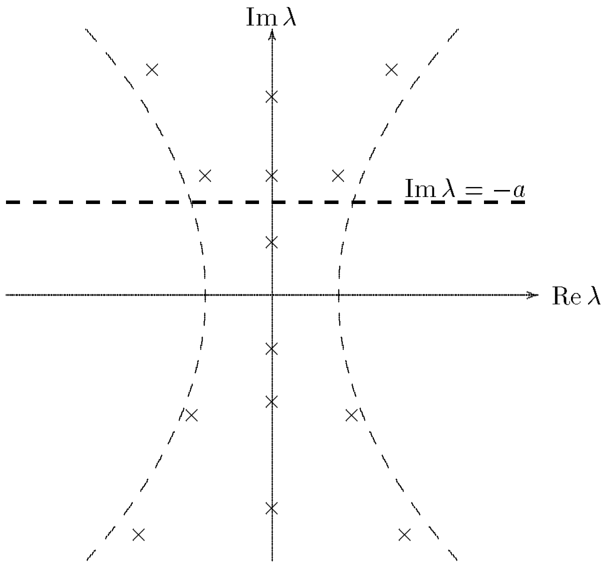

Figure 8. $\operatorname{spec}_b(P),\ a \notin -\operatorname{Im}\operatorname{spec}_b(P)$.

with poles of finite order and rank, with residues finite rank smoothing operators at points of $\operatorname{spec}_b(Q)$ *(the boundary spectrum) and furthermore*

$$\lambda_j \in \operatorname{spec}_b(Q),\ |\lambda_j| \longrightarrow \infty \Longrightarrow |\operatorname{Im}\lambda_j| \longrightarrow \infty.$$

In fact the proof will give slightly more than the statement in that continuity estimates on the inverse of the indicial family arise directly from the discussion. Nevertheless, Proposition 5.3 captures the most important fact, that the indicial family of an elliptic b-differential operator is invertible for almost all values (all except a discrete set) of λ. We shall also have occasion to use a more precise version of spec_b :

$$\text{(5.10)}\quad \operatorname{Spec}_b(Q) = \left\{(\lambda, k); \lambda \in \operatorname{spec}_b(Q) \text{ and } \widehat{Q}(z)^{-1} \text{ has a pole of order } k+1 \text{ at } \lambda\right\}.$$

To prove Proposition 5.3 it is useful to extend the notion of the indicial family to the small calculus of b-pseudodifferential operators. To do so (5.7) can first be interpreted in terms of the indicial operator of P, thought of as the restriction to the front face of the Schwartz kernel. Recall that the kernel, in the sense of (4.58), of the identity is given in local coordinates by

$$\kappa_{\mathrm{Id}} = \delta(s-1)\delta(y-y')|\frac{ds}{s}\frac{dr}{r}dydy'|^{\frac{1}{2}}. \tag{5.11}$$

Thus the indicial family arises by taking the Mellin transform in s after restricting to $\mathrm{bf}(X_b^2)$, i.e. $r = x + x' = 0$

$$I_\nu(\mathrm{Id}, \lambda) = \int_0^\infty s^{-i\lambda}\kappa(\mathrm{Id})_{|\mathrm{bf}}\frac{ds}{s} \equiv \mathrm{Id}\,. \tag{5.12}$$

For a general element of $\mathrm{Diff}_b^k(X; {}^b\Omega^{\frac{1}{2}})$ the kernel at the front face is just $I(P)$ applied to (5.11). From (5.12) it follows that

$$I_\nu(P, \lambda) = \int_0^\infty s^{-i\lambda}\kappa(P)_{|\mathrm{bf}}\frac{ds}{s}. \tag{5.13}$$

Thus the indicial family, with respect to ν, of elements of the small calculus can simply be *defined* by (5.13) and this is consistent with (5.6).

LEMMA 5.4. *The indicial family of $A \in \Psi_b^m(X; {}^b\Omega^{\frac{1}{2}})$ is an entire family*

$$\mathbb{C} \ni \lambda \longmapsto I_\nu(A, \lambda) \in \Psi^m(\partial X; \Omega^{\frac{1}{2}}) \tag{5.14}$$

which, if $m \leq 0$, satisfies the uniform continuity estimates:

$$\begin{gathered} I_\nu(A,\lambda)\colon L^2(\partial X; \Omega^{\frac{1}{2}}) \longrightarrow H^{-m}(\partial X; \Omega^{\frac{1}{2}}), \\ \|I_\nu(A,\lambda)\|_{0,-m} \leq C_{A,N},\ |\operatorname{Im}\lambda| \leq N \in \mathbb{N} \end{gathered} \tag{5.15}$$

$$\begin{gathered} I_\nu(A,\lambda)\colon L^2(\partial X; \Omega^{\frac{1}{2}}) \longrightarrow L^2(\partial X; \Omega^{\frac{1}{2}}), \\ \|I_\nu(A,\lambda)\|_{0,0} \leq C_{A,N}(1+|\lambda|)^m,\ |\operatorname{Im}\lambda| \leq N \in \mathbb{N}. \end{gathered} \tag{5.16}$$

EXERCISE 5.5. Write down (and prove) appropriate continuity estimates when $m \geq 0$.

PROOF: This follows from the representation (4.59) – (4.61) of the Schwartz kernel of an element $A \in \Psi^m(X; {}^b\Omega^{\frac{1}{2}})$. In (5.13) only the kernel at $\mathrm{bf}(X_b^2)$ is of interest. Away from the lifted diagonal, it is $\mathcal{C}^\infty$ and vanishes rapidly at the boundary. Thus, directly from Theorem 5.1 it follows that if $A \in \Psi_b^{-\infty}(X; {}^b\Omega^{\frac{1}{2}})$ then $I_\nu(A, \lambda)$ is an entire family of smoothing operators in $\Psi^{-\infty}(\partial X; \Omega^{\frac{1}{2}})$ with kernel rapidly decreasing, with all derivatives, as

$|\operatorname{Re}\lambda| \longrightarrow \infty$ in any region where $|\operatorname{Im}\lambda|$ is bounded. Such an operator certainly satisfies (5.15) and (5.16) for any m.

Thus it can be assumed that the kernel of A is supported in any small neighbourhood of the lifted diagonal and, by taking a partition of unity, in a coordinate patch. The representation (4.61) arises by taking as amplitude

$$b(r, y', \lambda, \eta) = \int e^{i\lambda \log s + i\eta\cdot(y-y')} k(r, s, y, y') \frac{ds}{s} dy'$$

since Δ_b is locally defined by $\log s = 0$, $y = y'$. Since k has support in s compact in $[0, \infty)$ it follows that $b(r, y', \lambda, \eta)$ is actually entire in λ and satisfies the symbol estimates (4.42), with $\xi = (\operatorname{Re}\lambda, \eta)$, uniformly in any strip where $|\operatorname{Im}\lambda|$ is bounded. In terms of the representation (4.61) the kernel of the indicial family is just

$$I_\nu(A, \lambda) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(y-y')\cdot\eta} b(0, y', \lambda, \eta) d\eta \tag{5.17}$$

so is indeed an entire family of pseudodifferential operators of order m on ∂X. The continuity estimates on an operator (5.17) only depend on the symbol estimates (and the $\mathcal{C}^\infty$ estimates on the kernel away from the diagonal) see [47]. If $m < 0$ the symbol estimates (4.42) on b :

$$|D^j_\lambda D^\alpha_{y'} D^\beta_\eta b(0, y', \lambda, \eta)| \le C_{j,\alpha,\beta,K}(1 + |\lambda| + |\eta|)^{m-j-|\beta|},\ y' \in K \subset\subset \mathbb{R}^n$$

give the uniform estimates

$$|D^\alpha_{y'} D^\beta_\eta b(0, y', \lambda, \eta)| \le C_{\alpha,\beta,K}(1 + |\eta|)^{m-|\beta|},\ y' \in K \subset\subset \mathbb{R}^n$$
$$|D^\alpha_{y'} D^\beta_\eta b(0, y', \lambda, \eta)| \le C_{\alpha,\beta,K}(1 + |\lambda|)^m (1 + |\eta|)^{-|\beta|},\ y' \in K \subset\subset \mathbb{R}^n.$$

To get the corresponding continuity estimates (5.15) and (5.16), one should treat $I_\nu(A, \lambda)$ as an element of the space $\Psi^m(\partial X, \Omega^{\frac12})$ and then as an element of $\Psi^0(\partial X; \Omega^{\frac12})$.

The indicial operators for the small calculus, given as the restriction of the kernels to the front face of X^2_b, satisfy

$$I(P \circ A) = I(P) \circ I(A),\ P \in \operatorname{Diff}^k_b(X; {}^b\Omega^{\frac12}). \tag{5.18}$$

From the definition, (5.13), of the indicial families it follows that

$$I_\nu(P \circ A, \lambda) = I_\nu(P, \lambda) \circ I_\nu(A, \lambda). \tag{5.19}$$

That is, the indicial family gives an entire family of maps

$$\Psi_b^m(X;{}^b\Omega^{\frac12}) \ni A \longmapsto I_\nu(A,\lambda) \in \Psi^m(\partial X;\Omega^{\frac12}),\ \lambda \in \mathbb{C} \tag{5.20}$$

which are multiplicative over $\mathrm{Diff}_b^*(X;{}^b\Omega^{\frac12})$. The maps (5.20) are actually algebra homomorphisms, a fact which will not be used for the moment but which allows the construction below of parametrices to be extended to elliptic elements of the small calculus.

Proof of Proposition 5.3: A parametrix for P in the small calculus was constructed in Chapter 4. Namely $G_s \in \Psi_b^{-k}(X;{}^b\Omega^{\frac12})$ is such that

$$P \circ G_s = \mathrm{Id} - R_s,\ R_s \in \Psi_b^{-\infty}(X;{}^b\Omega^{\frac12}).$$

Now, using (5.19) to pass to the indicial families, with respect to some boundary defining function,

$$I_\nu(P,\lambda) \circ I_\nu(G_s,\lambda) = \mathrm{Id} - I_\nu(R_s,\lambda). \tag{5.21}$$

The remainder term, $I_\nu(R_s,\lambda)$ is, by Lemma 5.62, an entire family of smoothing operators which are rapidly decreasing at real infinity. From (5.16) there is a real function $F : [0,\infty) \longrightarrow [0,\infty)$ such that $\mathrm{Id} - I_\nu(R_s,\lambda)$ is, as an operator on $L^2(\partial X;\Omega^{\frac12})$, invertible for $|\operatorname{Re}\lambda| \ge F(|\operatorname{Im}\lambda|)$. Set

$$(\mathrm{Id} - I_\nu(R_s,\lambda))^{-1} = \mathrm{Id} - S(\lambda),\ |\operatorname{Re}\lambda| \ge F(|\operatorname{Im}\lambda|). \tag{5.22}$$

Then

$$S(\lambda) = I_\nu(R_s,\lambda) + I_\nu(R_s,\lambda) \circ I_\nu(R_s,\lambda) + I_\nu(R_s,\lambda) \circ S(\lambda) \circ I_\nu(R_s,\lambda) \tag{5.23}$$

from which it follows that $S(\lambda)$ is also a smoothing operator depending holomorphically on λ in the region (5.22) and satisfying the same type of rapid decay estimates as $I_\nu(R,\lambda)$ in this region.

5.3. Analytic Fredholm theory.

In fact by analytic Fredholm theory it follows that $S(\lambda)$ extends to a meromorphic function on the whole of $\mathbb{C}$, with values in the smoothing operators. It is perhaps worthwhile to take the time to explain this well-known idea. First fix any $\lambda_0 \in \mathbb{C}$ at random. Since $I_\nu(R_s,\lambda_0)$ is a smoothing operator on a compact manifold the null space, N_0, of $\mathrm{Id} - I_\nu(R_s,\lambda_0)$ acting on $L^2(\partial X;\Omega^{\frac12})$ is finite dimensional and contained in $\mathcal{C}^\infty(\partial X;\Omega^{\frac12})$. Furthermore the range of $\mathrm{Id} - I_\nu(R_s,\lambda_0)$ is closed with a finite dimensional complement of the same dimension as N_0, $N_1 \subset \mathcal{C}^\infty(\partial X; \Omega^{\frac{1}{2}})$. Thus the operator $I_\nu(R_s, \lambda)$ can be split into a 2×2 matrix:

$$I_\nu(R_s, \lambda) = \begin{bmatrix} R_{00}(\lambda) & R_{01}(\lambda) \\ R_{10}(\lambda) & R_{11}(\lambda) \end{bmatrix}$$

acting on $L^2(\partial X; \Omega^{\frac{1}{2}}) = N \oplus N^\perp$, where N is a finite dimensional subspace of $\mathcal{C}^\infty(\partial X; \Omega^{\frac{1}{2}})$, $\mathrm{Id} - R_{11}(\lambda_0)$ is invertible and all the entries are analytic near λ_0. Then solving

$$[\mathrm{Id} - I_\nu(R_s, \lambda)]u = f \in L^2(\partial X; \Omega^{\frac{1}{2}}),\ f = f_N + f',\ u = u_N + u' \tag{5.24}$$

is equivalent to solving

$$[\mathrm{Id} - R_{00}(\lambda)]u_N + R_{01}(\lambda)u' = f_N,\ u_N + [\mathrm{Id} - R_{11}(\lambda)]u' = f'. \tag{5.25}$$

If λ is close to λ_0 then $\mathrm{Id} - R_{11}(\lambda)$ must still be invertible, so u' can be eliminated from (5.25) and the solvability of (5.24) reduces to that of

$$[\mathrm{Id} - R_{00} + R_{10} \circ (\mathrm{Id} - R_{11})^{-1}]u_N = f_N - R_{10} \circ (\mathrm{Id} - R_{11})^{-1} f'. \tag{5.26}$$

Here all the operators act on N, i.e. are finite matrices, depending holomorphically on λ in $|\lambda - \lambda_0| < \epsilon$. Thus (5.26) is always solvable if and only if the determinant

$$\det[\mathrm{Id} - R_{00} + R_{10} \circ (\mathrm{Id} - R_{11})^{-1}] \neq 0. \tag{5.27}$$

The set of λ for which $\mathrm{Id} - I_\nu(R_s, \lambda)$ is invertible near λ_0 is therefore exactly the set at which the holomorphic function in (5.27) does not vanish. Starting at the boundary of the set in (5.22) it follows that this must be a discrete set, i.e. the determinant in (5.27) does not vanish identically. From this discussion, and (5.23) as before, it follows that $S(\lambda)$ is indeed a meromorphic operator with finite rank residues at the points where it is singular. From (5.21) the inverse of $I_\nu(P, \lambda)$ is also meromorphic since it can be written

$$I_\nu(P, \lambda)^{-1} = I_\nu(G_s, \lambda) \circ (\mathrm{Id} - S(\lambda)).$$

The composition properties of pseudodifferential operators on ∂X show that the residues of the poles of $I_\nu(P, \lambda)^{-1}$ are also finite rank smoothing operators.

Not only does this complete the proof of Proposition 5.3 but it allows the inverse of the indicial family to be written

$$I_\nu(P, \lambda)^{-1} = I_\nu(G_s, \lambda) + R'(\lambda),\ G_s \in \Psi_b^{-k}(X; {}^b\Omega^{\frac{1}{2}}), \tag{5.28}$$

with $R'(\lambda)$ a meromorphic family of smoothing operators having residues of finite rank and decaying rapidly as $|\operatorname{Re} \lambda| \longrightarrow \infty$ in any region where $|\operatorname{Im} \lambda|$ is bounded.

EXERCISE 5.6. Extend Proposition 5.3 to give a similar description of the inverse of the indicial family of an elliptic element $Q \in \Psi^m_{b,I,\mathrm{os}}(X; {}^b\Omega^{\frac12})$.

5.4. Conjugation by powers.

There is a direct approach to the indicial family of a b-pseudodifferential operator, as in (5.20), which is mentioned in the Introduction. First observe that $\Psi^m_b(X; {}^b\Omega^{\frac12})$ is invariant under conjugation by complex powers of a boundary defining function.

PROPOSITION 5.7. *If $x \in \mathcal{C}^\infty(X)$ is a defining function for the boundary of X and $z \in \mathbb{C}$ then*

$$\Psi^m_b(X; {}^b\Omega^{\frac12}) \ni A \longmapsto x^{-z} A x^z \in \Psi^m_b(X; {}^b\Omega^{\frac12}) \tag{5.29}$$

is an isomorphism

PROOF: It is enough to show that the conjugated operator is in the space $\Psi^m_b(X; {}^b\Omega^{\frac12})$ as indicated, since the inverse map is obtained by replacing z by $-z$. The Schwartz kernel of $x^{-z}Ax^z$ is $(\frac{x}{x'})^{-z}K$, where K is the Schwartz kernel of A. Lifting to X^2_b, the kernel in the sense of (4.54) is just

$$(\frac{1-\tau}{1+\tau})^{-z}\kappa, \tag{5.30}$$

where κ is the kernel of A as in Definition 4.22. Since the multiplier here is $\mathcal{C}^\infty$ and non-vanishing in X^2_b, except at $\mathrm{lb} \cup \mathrm{rb}$, where however all derivatives have polynomial growth, the product (5.30) satisfies (4.59) - (4.61) too, i.e. (5.29) holds.

These observations can be used to obtain a result intermediate between Proposition 4.29 and Proposition 4.34:

COROLLARY. *If $k \in \mathbb{N}$ then each $A \in \Psi^*_b(X; {}^b\Omega^{\frac12})$ defines a continuous linear map*

$$A \colon x^k\mathcal{C}^\infty(X; {}^b\Omega^{\frac12}) \longrightarrow x^k\mathcal{C}^\infty(X; {}^b\Omega^{\frac12}).$$

This in turn allows (5.20) to be defined very simply when $\lambda = 0$. Namely if $\psi \in \mathcal{C}^\infty(\partial X; \Omega^{\frac12})$ then

$$A_\partial\psi = A\phi_{\restriction\partial X} \in \mathcal{C}^\infty(\partial X; \Omega^{\frac12}),\ \phi \in \mathcal{C}^\infty(X; {}^b\Omega^{\frac12}),\ \phi_{\restriction\partial X} = \psi \tag{5.31}$$

is independent of the choice of ϕ. The operator A_∂ defined by (5.31) is equal to $I_\nu(A, 0)$ for any choice of trivialization ν. This follows from the representation (4.68), which was extended to elements of $\mathcal{C}^\infty(X; {}^b\Omega^{\frac12})$ in the proof of Proposition 4.34. Restricting (4.68) to $x = 0$ shows that the kernel of A_∂ is just

$$\int_0^\infty \kappa(0, s, y, y') \frac{ds}{s}.$$

This corresponds to (5.13) for $\lambda = 0$ as claimed.

More generally the indicial family of a b-pseudodifferential operator can also be determined in this way:

PROPOSITION 5.8. *If $x \in \mathcal{C}^\infty(X)$ is a defining function for the boundary and $dx \cdot \nu = 1$ then the indicial family, defined by* (5.13) *is also given by*

$$I_\nu(A, \lambda) = [x^{-i\lambda} A x^{i\lambda}]_\partial, \tag{5.32}$$

where the boundary operator is fixed by (5.31).

5.5. Commutator identity for the b-trace.

Now that the indicial family of a b-pseudodifferential operator has been introduced in general, the formula for the b-trace of a commutator mentioned in §4.20 can be deduced.

PROPOSITION 5.9. *If $A, B \in \Psi_b^{-\infty}(X; {}^b\Omega^{\frac{1}{2}})$ then*

$$\text{b-Tr}_\nu([A, B]) = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \operatorname{tr}\left(\partial_\lambda I_\nu(A, \lambda) \circ I_\nu(B, \lambda)\right) d\lambda. \tag{5.33}$$

PROOF: It suffices to prove (5.33) for the elements of a subset the span of which is dense in $\Psi_b^{-\infty}(X; {}^b\Omega^{\frac{1}{2}})$. Thus it can be assumed that both A and B have kernels with small supports in X_b^2 and disjoint from $\operatorname{lb}(X_b^2)$ and $\operatorname{rb}(X_b^2)$ (but *not* $\operatorname{bf}(X_b^2)$ since this would not give a dense subset). Consider first the product $A \circ B$. The b-trace is, by definition,

$$\text{b-Tr}_\nu(A \circ B) = \lim_{\epsilon \downarrow 0} \left\{ \int_{x > \epsilon} (AB)_{\restriction \Delta} + \log \epsilon \cdot \gamma \right\}$$

for the appropriate constant γ; here the kernels are again denoted as the operators. In $x > \epsilon$ the kernel is given by a convergent integral (since no points near the corner of X^2 are involved):

$$\int_{x > \epsilon} (AB)_{\restriction \Delta} = \int_{x > \epsilon} \left(\int_X A(z, z') B(z', z) \right). \tag{5.34}$$

Let $R\colon X^2 \longleftrightarrow X^2$ be the factor-exchanging isomorphism, $R(z,z') = (z',z)$. Thus R^*A is the kernel of A^t. Then (5.34) can be written

$$\int\limits_{x>\epsilon} (AB)_{\restriction\Delta} = \int\limits_{\{x>\epsilon\}\cap X^2} A\cdot R^*B. \tag{5.35}$$

Here x is the boundary defining function lifted to the left factor. Since $x=0$ at $\mathrm{bf}(X_b^2)$, the lift of (5.35) to X_b^2 is trivial and

$$\int\limits_{x>\epsilon} (AB)_{\restriction\Delta} = \int\limits_{\{x>\epsilon\}\cap X_b^2} A\cdot R^*B. \tag{5.36}$$

Of course the same discussion applies to $B\circ A$. Since R lifts to an isomorphism of X_b^2 exchanging x and x', the lift of the defining function from the right factor gives

$$\int\limits_{x>\epsilon} (BA)_{\restriction\Delta} = \int\limits_{\{x>\epsilon\}\cap X_b^2} B\cdot R^*A = \int\limits_{\{x'>\epsilon\}\cap X_b^2} A\cdot R^*B. \tag{5.37}$$

The integrands in (5.36) and (5.37) are the same, only the domains differ. To complete the computation introduce local coordinates (as can certainly be done by assuming that the supports of the kernels are small). The supports do not meet $\mathrm{lb}(X_b^2)$ or $\mathrm{rb}(X_b^2)$, so projective coordinates can be used:

$$x,\ s=\frac{x}{x'},\ y,\ y',\ A=\alpha\left|\frac{ds}{s}\frac{dx}{x}dydy'\right|^{\frac12},\ B=\beta\left|\frac{ds}{s}\frac{dx}{x}dydy'\right|^{\frac12}.$$

Then the involution R is $(x,s,y,y') \longleftrightarrow (\frac{x}{s},1/s,y',y)$. From (5.36) and (5.37) it follows that

$$\begin{aligned}&\int\limits_{x>\epsilon} (AB-BA)_{\restriction\Delta}\\ &= \int\limits_{\partial X\times\partial X}\int\limits_0^\infty\int\limits_\epsilon^{\epsilon s} \alpha(x,s,y,y')\beta\left(\frac{x}{s},1/s,y',y\right)\frac{dx}{x}\frac{ds}{s}dydy'.\end{aligned}$$

Here α,β have compact support in $[0,\infty)$ in s. Replacing α and β by their values at $x=0$ gives integrable errors which therefore disappear in the limit $\epsilon \downarrow 0$, so

$$\begin{aligned}
&\lim_{\epsilon\downarrow 0} \int\limits_{x>\epsilon} (K_{AB} - K_{BA})_{\restriction\Delta} \\
&= \lim_{\epsilon\downarrow 0} \int\limits_{\partial X\times\partial X} \int\limits_0^\infty \int\limits_\epsilon^{\epsilon s} \alpha(0,s,y,y')\beta(0,\frac{1}{s},y',y)\frac{dx}{x}\frac{ds}{s}dydy' \qquad (5.38)\\
&= \int\limits_{\partial X\times\partial X} \int\limits_0^\infty \log s\cdot\alpha(0,s,y,y')\beta(0,\frac{1}{s},y',y)\frac{ds}{s}dydy'
\end{aligned}$$

exists, without regularization.

The kernels $\alpha(0,s,y,y')|\frac{ds}{s}dydy'|^{\frac{1}{2}}$ and $\beta(0,s,y,y')|\frac{ds}{s}dydy'|^{\frac{1}{2}}$ are, by definition, those of the indicial operators. Using the inverse of the Mellin transform they can be written in terms of the kernels of the indicial families as

$$\begin{aligned}
\alpha(0,s,y,y')|\frac{ds}{s}dydy'|^{\frac{1}{2}} &= \frac{1}{2\pi}\int s^{i\lambda} I_\nu(A,\lambda)(y,y')d\lambda|\frac{ds}{s}|^{\frac{1}{2}}\\
\overline{\beta(0,\frac{1}{s},y,y')}|\frac{ds}{s}dydy'|^{\frac{1}{2}} &= \frac{1}{2\pi}\int s^{i\lambda}\overline{I_\nu(B,\lambda)}(y,y')d\lambda|\frac{ds}{s}|^{\frac{1}{2}}.
\end{aligned}$$

The Plancherel formula for the Mellin (i.e. Fourier) transform combined with integration by parts, shows that

$$(\log s)\alpha(0,s,y,y')|\frac{ds}{s}dydy'| = \frac{i}{2\pi}\int s^{i\lambda}\partial_\lambda I_\nu(A,\lambda)(y,y')d\lambda|\frac{ds}{s}|^{\frac{1}{2}},$$

and so allows (5.38) to be rewritten as

$$\begin{aligned}
&\lim_{\epsilon\downarrow 0} \int\limits_{x>\epsilon} (K_{AB} - K_{BA})_{\restriction\Delta} \qquad (5.39)\\
&= -\frac{1}{2\pi i}\int\limits_{\partial X\times\partial X}\int\limits_{-\infty}^{\infty} \partial_\lambda I_\nu(A,\lambda)(y,y')\cdot I_\nu(B,\lambda)(y',y)d\lambda.
\end{aligned}$$

The integrals over ∂X just give the trace of the composite operator, so this is precisely (5.33).

In fact it is most important to have (5.33) when one of the operators, say A, is a b-differential operator. This is easily proved directly (see Exercise 5.11) but at this stage it is perhaps more enlightening to deduce it from (5.33).

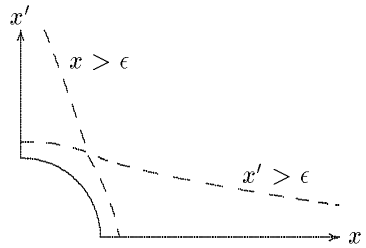

Figure 9. The domains in X_b^2 for (5.34), (5.35).

LEMMA 5.10. *The identity* (5.33) *holds if* $A \in \mathrm{Diff}_b^m(X; {}^b\Omega^{\frac{1}{2}})$ *and* $B \in \Psi_b^{-\infty}(X; {}^b\Omega^{\frac{1}{2}})$.

PROOF: In fact, by continuity, (5.33) extends to allow $A \in \Psi_b^*(X; {}^b\Omega^{\frac{1}{2}})$ if $B \in \Psi_b^{-\infty}(X; {}^b\Omega^{\frac{1}{2}})$.

EXERCISE 5.11. Give a direct proof of Lemma 5.10 by lifting the operator $A \in \mathrm{Diff}_b^m(X; {}^b\Omega^{\frac{1}{2}})$ to an operator $\tilde{A} \in \mathrm{Diff}_b^m(X_b^2; {}^b\Omega^{\frac{1}{2}})$ from the left factor and $\tilde{A}^t \in \mathrm{Diff}_b^m(X_b^2; {}^b\Omega^{\frac{1}{2}})$ from the right factor and observing that the kernel of $[A, B]$ is

$$\tilde{A} \cdot K_B - \tilde{A}^t \cdot K_B \quad \text{on } X_b^2.$$

Restrict this to the (lifted) diagonal, in $x > \epsilon$, and integrate to get the left side of (5.39) in this case. Now (for example working in local coordinates by assuming that B has small support) show that integration by parts reduces the integral over $x > \epsilon$ to one over $x = \epsilon$ which, in the limit as $\epsilon \downarrow 0$, becomes an integral over $\mathrm{bf}(X_b^2)$. With a little manipulation this integral can be brought to the form of the right side of (5.33).

5.6. Invertibility of the indicial operator.

The discussion above of the invertibility properties of the indicial family of an elliptic b-pseudodifferential operator on a compact manifold with boundary can be used to examine the mapping and invertibility properties of the indicial operator itself. This in turn is the main step in obtaining an understanding of the invertibility properties of the original operator.

Recall that the indicial family of $P \in \Psi_b^m(X; {}^b\Omega^{\frac{1}{2}})$ is defined, in (5.13), by taking the Mellin transform of the indicial operator, which is fixed by the kernel of P restricted to the front face of X_b^2. To show the implications of this for $I(P)$, first consider some Sobolev spaces on which it should act.

The Mellin transform (5.1) extends by continuity to an isomorphism

$$L^2\left(\widetilde{X};{}^b\Omega^{\frac{1}{2}}\right) \overset{()_{M,\nu}}{\longleftrightarrow} L^2\left(\mathbb{R}\times\partial X;|d\lambda|^{\frac{1}{2}}\cdot\Omega^{\frac{1}{2}}\right). \tag{5.40}$$

Indeed this is just Plancherel's theorem applied to (5.4). This isomorphism can be used to define the b-Sobolev spaces of any order:

(5.41)

$$H_b^m(\widetilde{X};{}^b\Omega^{\frac{1}{2}}) = \Big\{u\in L^2(\widetilde{X};{}^b\Omega^{\frac{1}{2}}); u_{M,\nu}\in L^2(\mathbb{R};H^m(\partial X;\Omega^{\frac{1}{2}})), \\ (1+|\lambda|^2)^{m/2}u_{M,\nu}\in L^2(\mathbb{R}\times\partial X;|d\lambda|^{\frac{1}{2}}\Omega^{\frac{1}{2}})\Big\},\ m\ge 0.$$

EXERCISE 5.12. Show that the space defined by (5.41) is independent of the section ν used to define the Mellin transform.

In fact this space is actually invariant under arbitrary diffeomorphisms of $\widetilde{X}$ as a compact manifold with boundary and so can be transferred to X itself. For present purposes it suffices to see this when $m\in\mathbb{Z}$. When $m\in\mathbb{N}$, (5.41) can be directly formulated on an arbitrary compact manifold with boundary as

(5.42)

$$H_b^m(X;{}^b\Omega^{\frac{1}{2}}) = \{u\in L^2(X;{}^b\Omega^{\frac{1}{2}}); \\ \operatorname{Diff}_b^m(X;{}^b\Omega^{\frac{1}{2}})u\subset L^2(X;{}^b\Omega^{\frac{1}{2}})\},\ m\in\mathbb{N},$$

and the coordinate-invariance follows from that of $L^2(X;{}^b\Omega^{\frac{1}{2}})$.

It is convenient to have at hand the negative order Sobolev spaces as well. For integral orders a similar, but dual, definition to (5.42) can be used:

(5.43)

$$H_b^m(X;{}^b\Omega^{\frac{1}{2}}) = L^2(X;{}^b\Omega^{\frac{1}{2}}) + \operatorname{Diff}_b^{-m}(X;{}^b\Omega^{\frac{1}{2}})\cdot L^2(X;{}^b\Omega^{\frac{1}{2}}),\ m\in-\mathbb{N}.$$

The general case will be discussed later using the continuity properties of b-pseudodifferential operators.

The elementary considerations of Chapter 1 underscore the importance of considering *weighted* versions of spaces of this type. Let $\rho_i\in\mathcal{C}^\infty(X)$ for $i=1,\dots,J$ be defining functions for the bounding hypersurfaces (i.e.

boundary components) of a compact manifold with boundary and for $\mathfrak{a} = (\alpha_1, \dots, \alpha_J) \in \mathbb{R}^J$ set

$$\rho^{\mathfrak{a}} H^m_b(X; {}^b\Omega^{\frac12}) = \Big\{ u \in \mathcal{C}^{-\infty}(X; {}^b\Omega^{\frac12}); \tag{5.44}$$
$$u = \rho_1^{\alpha_1} \cdots \rho_J^{\alpha_J} v, v \in H^m_b(X; {}^b\Omega^{\frac12}) \Big\},\ m \in \mathbb{Z}.$$

EXERCISE 5.13. There is a choice of ordering involved in (5.44) since, for $m > 0$, it says exactly:

$$u \in \rho^{\mathfrak{a}} H^m_b(X; {}^b\Omega^{\frac12}) \Longleftrightarrow \operatorname{Diff}^m_b(X; {}^b\Omega^{\frac12})[\rho^{-\mathfrak{a}} u] \subset L^2(X; {}^b\Omega^{\frac12}),$$

where ρ^{α} is written for $\rho_1^{\alpha_1} \cdots \rho_J^{\alpha_J}$. Show that in fact the order is immaterial and

$$u \in \rho^{\mathfrak{a}} H^m_b(X; {}^b\Omega^{\frac12}) \Longleftrightarrow$$
$$\operatorname{Diff}^m_b(X; {}^b\Omega^{\frac12}) u \subset \rho^{\mathfrak{a}} L^2(X; {}^b\Omega^{\frac12}) = \rho^{\mathfrak{a}} H^0_b(X; {}^b\Omega^{\frac12}).$$

If $m < 0$ show that similarly (5.44) is just

$$u \in \rho^{\mathfrak{a}} H^m_b \Longleftrightarrow$$
$$\rho^{-\mathfrak{a}} u \in L^2(X; {}^b\Omega^{\frac12}) + \operatorname{Diff}^{-m}_b(X; {}^b\Omega^{\frac12}) \cdot L^2(X; {}^b\Omega^{\frac12}),\ m \in -\mathbb{Z},$$

and that alternatively

$$u \in \rho^{\mathfrak{a}} H^m_b \Longleftrightarrow$$
$$u \in \rho^{\mathfrak{a}} L^2(X; {}^b\Omega^{\frac12}) + \operatorname{Diff}^{-m}_b(X; {}^b\Omega^{\frac12}) \cdot \rho^{\mathfrak{a}} L^2(X; {}^b\Omega^{\frac12}),\ m \in -\mathbb{Z}.$$

As is to be expected b-differential operators are bounded on these spaces:

LEMMA 5.14. *For any $k \in \mathbb{N}$ each $P \in \operatorname{Diff}^k_b(X; {}^b\Omega^{\frac12})$ defines a continuous linear map*

$$P : \rho^{\mathfrak{a}} H^m_b(X; {}^b\Omega^{\frac12}) \longrightarrow \rho^{\mathfrak{a}} H^{m-k}_b(X; {}^b\Omega^{\frac12}) \tag{5.45}$$

for any $m \in \mathbb{Z}$ and any $\mathfrak{a} \in \mathbb{R}^J$ where X has J boundary components.

PROOF: If $\mathfrak{a} \in \mathbb{R}^J$ then conjugation by $\rho^{\mathfrak{a}}$ gives an isomorphism (as in (5.29))

$$\operatorname{Diff}^k_b(X; {}^b\Omega^{\frac12}) \ni P \longmapsto \rho^{\mathfrak{a}} P \rho^{-\mathfrak{a}} \in \operatorname{Diff}^k_b(X; {}^b\Omega^{\frac12}).$$

Thus it suffices to consider the case $\mathfrak{a} = 0$. First suppose $m \geq k$. Then if $u \in H^m_b(X; {}^b\Omega^{\frac12})$ and $Q \in \operatorname{Diff}^{m-k}_b(X; {}^b\Omega^{\frac12})$ certainly $Q \circ P \in \operatorname{Diff}^m_b(X; {}^b\Omega^{\frac12})$, so $Q(Pu) \in L^2(X; {}^b\Omega^{\frac{1}{2}})$ and hence $Pu \in H_b^{m-k}(X; {}^b\Omega^{\frac{1}{2}})$. Continuity is immediate from the same argument, so (5.45) follows. The case $m < 0$ is similar and if $0 \le m \le k$ then it is enough to use (2.20) to decompose P into a finite sum

$$P = \sum_r P_r Q_r,\ P_r \in \mathrm{Diff}^{k-m}(X; {}^b\Omega^{\frac{1}{2}}),\ Q_r \in \mathrm{Diff}_b^m(X; {}^b\Omega^{\frac{1}{2}})$$

since then each $Q_r u \in L^2(X; {}^b\Omega^{\frac{1}{2}})$, so

$$Pu = \sum_r P_r(Q_r u) \in H_b^{m-k}(X; {}^b\Omega^{\frac{1}{2}}).$$

Recall that the boundary of the 'model' space, $\widetilde{X}$, is divided into two parts. Each is isomorphic to ∂X, which will be assumed connected to reduce the notational overhead. Let respective defining functions be ρ_0 and ρ_∞, corresponding to the zero section of $N\partial X$ and infinity in this vector bundle. If $\alpha \in \mathbb{R}$ let $\rho^\alpha H_b^k(\widetilde{X}; {}^b\Omega^{\frac{1}{2}})$ denote the space with weighting factor $\rho_0^\alpha \rho_\infty^{-\alpha}$, with 'opposite' weighing at the two boundary faces. From a functional analytic point of view the main result concerning indicial operators is:

PROPOSITION 5.15. *If $Q \in \mathrm{Diff}_{b,I}^k(\widetilde{X}; {}^b\Omega^{\frac{1}{2}})$ is elliptic then*

$$Q : \rho^\alpha H_b^m(\widetilde{X}; {}^b\Omega^{\frac{1}{2}}) \longleftrightarrow \rho^\alpha H_b^{m-k}(\widetilde{X}; {}^b\Omega^{\frac{1}{2}}) \text{ is an isomorphism} \tag{5.46}$$

for some (and equivalently all) m if and only if

$$\alpha \notin -\operatorname{Im}\operatorname{spec}_b(Q) \text{ i.e. } \alpha \in \mathbb{R},\ \alpha \neq -\operatorname{Im}\lambda\ \forall\ \lambda \in \operatorname{spec}_b(Q). \tag{5.47}$$

PROOF: By conjugation it suffices to consider the case $\alpha = 0$; for simplicity suppose also that $m = k$. Thus, under the assumption

$$\operatorname{spec}_b(Q) \cap \mathbb{R} = \emptyset \tag{5.48}$$

it needs to be shown that Q is an isomorphism from $H_b^m(\widetilde{X}; {}^b\Omega^{\frac{1}{2}})$ to $L^2(\widetilde{X}; {}^b\Omega^{\frac{1}{2}})$. From (5.6) it follows that, after Mellin transformation, Q acts as the indicial family, $Q(\lambda)$. Combining (5.40) and (5.41) it suffices to show that

$$\begin{aligned} v \in L^2(\mathbb{R}\times\partial X; |d\lambda|^{\frac{1}{2}}\cdot\Omega^{\frac{1}{2}}) \Longrightarrow& \\ (1+|\lambda|^2)^{\frac{k}{2}} Q(\lambda)^{-1} v \in& L^2(\mathbb{R}\times\partial X; |d\lambda|^{\frac{1}{2}}\cdot\Omega^{\frac{1}{2}}), \\ Q(\lambda)^{-1} v \in& L^2(\mathbb{R}; H^m(\partial X; \Omega^{\frac{1}{2}})). \end{aligned} \tag{5.49}$$

Consider the decomposition of the inverse of the indicial family given by (5.28). The first term is the indicial family of an element of $\Psi^{-k}_{b,\mathrm{os}}(X; {}^b\Omega^{\frac12})$, so for this part the analogues of (5.49) follow directly from (5.15) and (5.16). The second term in (5.28) is non-singular precisely because of (5.48) and consists therefore of a family of smoothing operators on ∂X, depending smoothly on λ and decreasing rapidly as $|\lambda| \longrightarrow \infty$. This proves the proposition when $k = m$. The general case is similar.

5.7. Kernel of the inverse of the indicial operator.

Although this is quite satisfactory from a functional analytic point of view, it is important to see the specific nature of the Schwartz kernel of the inverse of Q i. e. to consider the action of Q^{-1} on Dirac delta 'functions.' In (5.13) and (5.28) the inverse of Q is presented as the sum of a term in the small calculus (the parametrix constructed in Chapter 4) and the inverse Mellin transform of the second term in (5.28). So consider the smooth kernel, reverting to compact coordinates:

$$(5.50)\quad R'_B(\tau, y, y') = \frac{1}{2\pi} \int\limits_{\mathrm{Im}\,\lambda = -\alpha} \left(\frac{1+\tau}{1-\tau}\right)^{i\lambda} R'(\lambda, y, y') d\lambda |dydy'|^{\frac12} \left|\frac{dx}{x}\right|^{\frac12},$$

with (5.47) assumed so that the integral is defined.

LEMMA 5.16. *For any $\alpha \notin -\mathrm{Im}\,\mathrm{spec}_b(Q)$ where $Q \in \mathrm{Diff}^k_{b,I}(\widetilde{X}; {}^b\Omega^{\frac12})$ is elliptic, the integral* (5.50)*, with R' as in* (5.28)*, gives $R'_B \in \mathcal{C}^\infty((-1,1) \times (\partial X)^2; {}^b\Omega^{\frac12})$ having asymptotic expansions at the boundaries $\tau = \pm 1$:*

$$(5.51)\qquad R'_B(\tau, y, y') \sim \sum_{\substack{z \in \mathrm{spec}_b(Q) \\ k \le \mathrm{ord}(z) \\ \pm\,\mathrm{Im}\, z > \mp \alpha}} \left(\frac{1+\tau}{1-\tau}\right)^{iz} \left[\log\left(\frac{1+\tau}{1-\tau}\right)\right]^k A_{z,k}(y, y'),$$

where the coefficients $A_{z,k}(y, y')$ are finite rank smoothing operators.

PROOF: Recall the meaning of the asymptotic expansion in (5.51). If $\phi \in \mathcal{C}^\infty(\mathbb{R})$ takes the value 1 near 1 and vanishes near -1 then (5.51) just means that for any $N \in \mathbb{N}$

$$(5.52)\qquad \phi(\pm\tau)\Bigg[R'_B(\tau, y, y') - \sum_{N \ge \pm\,\mathrm{Im}\, z > \mp\alpha} \left(\frac{1+\tau}{1-\tau}\right)^{iz} \left[\log\left(\frac{1+\tau}{1-\tau}\right)\right]^k A_{z,k}(y, y') \Bigg] \in \dot{\mathcal{C}}^N([-1,1] \times (\partial X)^2),$$

i.e. the difference must be N times continuously differentiable with all N derivatives vanishing at the boundary. The sum in (5.52) is still limited to $z \in \operatorname{spec}_b(Q)$ and $k \leq \operatorname{ord}(z)$. Of course the regularity statement is really two separate statements, near $\tau = 1$ and $\tau = -1$. It is simpler to rewrite (5.52) in terms of s near $\tau = -1$, corresponding to $s = 0$.

Thus it needs to be shown that

(5.53)
$$R'_B(s, y, y') = \frac{1}{2\pi} \int\limits_{\operatorname{Im} \lambda = -\alpha} s^{i\lambda} R'(\lambda, y, y') \sim \sum_{\substack{z \in \operatorname{spec}_b(Q) \\ k \leq \operatorname{ord}(z) \\ \operatorname{Im} z < -\alpha}} s^{iz} [\log(s)]^k A_{z,k}(y, y'), \text{ as } s \downarrow 0.$$

Simply estimating the integral near $s = 0$ gives

$$\left| D_s^p D_{y,y'}^\delta R'_B(s, y, y') \right| \leq C|s|^{\alpha - p}, \ s < 1.$$

If $\alpha > N$ then $R'_B \in \dot{\mathcal{C}}^N([0, 1) \times (\partial X)^2)$. Since the integrand in (5.53) is rapidly decreasing at real infinity, the contour can always be moved from $\operatorname{Im} \lambda = -\alpha$ to $\operatorname{Im} \lambda = -N - \epsilon$, $\epsilon > 0$, and hence ensure this regularity, except that by Cauchy's formula the residues at all the poles

$$z \in \operatorname{spec}_b(Q), \ -\alpha > \operatorname{Im} z \geq -N$$

need to be added. These residues give precisely the sum in (5.53), by virtue of the properties of $R'(\lambda)$ in (5.28). This shows the existence and form of the expansion near $s = 0$, i.e. $\tau = -1$. Changing variable in the integral from λ to $-\lambda$ and replacing s by $1/s$, the same argument can be applied to give the expansion at $\tau = 1$, i.e. $s = \infty$. This proves the proposition.

Notice that the expansion at $\operatorname{lb}(X_b^2)$, corresponding to $\tau = -1$ arises from the poles of $R'(\lambda)$, i.e the points of $\operatorname{spec}_b(Q)$, in $\operatorname{Im} \lambda < -\alpha$, whereas the expansion near $\operatorname{rb}(X_b^2)$ corresponds to the poles in $\operatorname{Im} \lambda > -\alpha$. One can think of the choice of weight, i.e. the choice of $\alpha \in \mathbb{R}$, as splitting the λ-plane in which $\operatorname{spec}_b(Q)$ is defined, into the upper part, corresponding to rb and the lower part, corresponding to lb.

5.8. Index formula for invariant operators.

Using these results on the indicial operator a simple index formula in the $\mathbb{R}^+$-invariant case can be derived. It is quite analogous to that for the half line, as described in Chapter 1. Consider the space $\widetilde{X}$ and let ρ_0 and ρ_∞ be defining functions for the zero section and ∞.

PROPOSITION 5.17. *Let $Q \in \mathrm{Diff}^m_{b,I}(\widetilde{X};{}^b\Omega^{\frac{1}{2}})$ be an elliptic differential operator and suppose $\alpha,\ \beta \in \mathbb{R}$ satisfy*

$$\alpha \notin -\operatorname{Im}\operatorname{spec}_b(Q),\ \ \beta \notin \operatorname{Im}\operatorname{spec}_b(Q) \tag{5.54}$$

then

$$\begin{gathered}\beta > -\alpha \Longrightarrow Q : \rho_0^\alpha\rho_\infty^\beta H_b^k(\widetilde{X};{}^b\Omega^{\frac{1}{2}}) \longrightarrow \rho_0^\alpha\rho_\infty^\beta H_b^{k-m}(\widetilde{X};{}^b\Omega^{\frac{1}{2}}) \\ \textit{is injective with closed range of codimension} \sum_{\substack{z\in\operatorname{spec}_b(Q)\\ \operatorname{Im} z\in[-\alpha,\beta]}} \operatorname{rank}(z)\end{gathered} \tag{5.55}$$

and

$$\begin{gathered}\beta < -\alpha \Longrightarrow Q : \rho_0^\alpha\rho_\infty^\beta H_b^k(\widetilde{X};{}^b\Omega^{\frac{1}{2}}) \twoheadrightarrow \rho_0^\alpha\rho_\infty^\beta H_b^{k-m}(\widetilde{X};{}^b\Omega^{\frac{1}{2}}) \\ \textit{is surjective with null space of dimension} \sum_{\substack{z\in\operatorname{spec}_b(Q)\\ \operatorname{Im} z\in[\beta,-\alpha]}} \operatorname{rank}(z).\end{gathered} \tag{5.56}$$

In particular the index is, up to sign, the sum of the ranks of the points in $\operatorname{spec}_b(Q)$ with imaginary parts between $-\alpha$ and β :

$$\operatorname{ind}(Q) = \operatorname{sgn}(\alpha+\beta) \sum_{\substack{z\in\operatorname{spec}_b(Q)\\ \operatorname{Im} z\in[-\alpha,\beta]\cup[\beta,-\alpha]}} \operatorname{rank}(z).$$

PROOF: Consider (5.55) first. Since $\beta > -\alpha$ the weighting at infinity is 'smaller' than that at zero in the sense that

$$\rho_0^\alpha\rho_\infty^\beta H_b^k(\widetilde{X};{}^b\Omega^{\frac{1}{2}}) \subset \rho^\alpha H_b^k(\widetilde{X};{}^b\Omega^{\frac{1}{2}}),$$

where the space on the right is the one that appears in Proposition 5.15. Thus the injectivity follows from (5.46). It remains then to show that the range is closed and to compute its codimension. As in the proof of Proposition 5.15, the Mellin transform and the inverse of the indicial family can be used. To do so a characterization of the range of the Mellin transform on the mixed-weight space is required. This is again a standard result of Paley-Wiener type for the Fourier transform:

LEMMA 5.18. *If $\alpha+\beta > 0$ then the Mellin transform gives an isomorphism*

$$\begin{gathered}\rho_0^\alpha\rho_\infty^\beta L^2(\widetilde{X};{}^b\Omega^{\frac{1}{2}}) \longrightarrow \Big\{u_M : \mathbb{R}\times i(-\alpha,\beta) \ni \lambda \longmapsto u_M(\lambda) \in L^2(\partial X;\Omega^{\frac{1}{2}}); \\ u_M(\lambda) \textit{ is holomorphic in } \lambda \textit{ and } \sup_{-\alpha<r<\beta} \|u_M(\cdot+ir,\cdot)\|_{L^2} < \infty\Big\}.\end{gathered} \tag{5.57}$$

A similar result can easily be deduced for the Mellin transforms of the weighted Sobolev spaces. Thus the range of (5.55) can be found, for simplicity when $m = k$. Namely it consists of those elements in the range of (5.57) which are of the form $u_M = Q(\lambda)v_M$ where $v \in \rho_0^\alpha \rho_\infty^\beta H_b^m(\widetilde{X}; {}^b\Omega^{\frac{1}{2}})$. This means that

$$v_M(\lambda) = Q(\lambda)^{-1}u_M(\lambda) \text{ is holomorphic in } -\alpha < \operatorname{Im}\lambda < \beta \tag{5.58}$$

since the regularity follows directly from the mapping properties of $Q(\lambda)^{-1}$ as before. In fact (5.58) means that all the singular terms in $Q(\lambda)^{-1}u_M(\lambda)$ must vanish. At each point $\lambda \in \operatorname{spec}_b(Q)$ with $-\alpha < \operatorname{Im}\lambda < \beta$ this imposes, by definition, exactly $\operatorname{rank}(\lambda)$ linearly independent conditions on $u_M(\lambda)$. The conditions for different points in $\operatorname{spec}_b(Q)$ are independent. To see this it is enough to show that there exists u_M having a given finite Taylor series at each of a finite number of points λ_i, which in turn follows by superposition with polynomial coefficients, since one can always choose $u \in \dot{\mathcal{C}}^\infty([-1,1])$ with u_M non-zero at a given point. So the range has the finite codimension given in (5.55), and is closed.

Note that it could not be concluded that the range was closed if (5.54) did not hold, and in fact it is not closed unless these conditions hold.

The other case, (5.56), can be analyzed by duality. The details are left as an exercise. However, note that the null space in (5.56) can be computed precisely in terms of the generalized null spaces of $Q(\lambda)$ for the appropriate elements of $\operatorname{spec}_b(Q)$:

$$\begin{gathered}\Big\{u \in \rho_0^\alpha \rho_\infty^\beta H_b^k(\widetilde{X}; {}^b\Omega^{\frac{1}{2}}); Qu = 0\Big\} = \\ \Big\{u = \sum_{\substack{z \in \operatorname{spec}_b(Q),\ k \le \operatorname{ord}(z) \\ -\alpha < \operatorname{Im} z < -\beta}} s^{iz}(\log s)^k u_{z,k}; \\ \sum_{p=r}^{k} D_z^{p-r} Q(z) u_{z,p} = 0,\ 0 \le r \le k\Big\}.\end{gathered} \tag{5.59}$$

Moreover the range of Q in (5.55) can be written

$$\begin{aligned}\big\{f \in \rho_0^\alpha \rho_\infty^\beta H_b^{k-m}(\widetilde{X}; {}^b\Omega^{\frac{1}{2}}); v(f) = 0 \\ \forall\ v \in \rho_0^{-\alpha}\rho_\infty^{-\beta} H_b^\infty(\widetilde{X}, {}^b\Omega^{\frac{1}{2}}) \text{ satisfying } Q^t v = 0\big\}.\end{aligned}$$

Here Q^t is the transpose with respect to the intrinsic (real) pairing between the spaces $\rho_0^\alpha \rho_\infty^\beta H_b^k(\widetilde{X}; {}^b\Omega^{\frac{1}{2}})$ and $\rho_0^{-\alpha}\rho_\infty^{-\beta} H_b^{-k}(\widetilde{X}; {}^b\Omega^{\frac{1}{2}})$. Thus (5.59) can be used to express these linear constraints in a precise form.

EXERCISE 5.19. Extend Proposition 5.17 to the pseudodifferential case, i.e. to elliptic elements of $\Psi^m_{b,I,\mathrm{os}}(\widetilde{X}, {}^b\Omega^{\frac12})$.

5.9. Composition in the small calculus.

An argument will now be given which proves, using the standard composition properties of pseudodifferential operators on compact manifolds without boundary, the composition properties for b-pseudodifferential operators already hinted at. It should noted, again, that there is a somewhat 'better' proof (see for example [63]) which is more geometric and which generalizes more readily.[7]

PROPOSITION 5.20. *If X is a compact manifold with boundary then*

$$\Psi^m_b(X; {}^b\Omega^{\frac12}) \circ \Psi^{m'}_b(X; {}^b\Omega^{\frac12}) \subset \Psi^{m+m'}_b(X; {}^b\Omega^{\frac12}).$$

PROOF: First it can be assumed that the operators, $A \in \Psi^m_b(X; {}^b\Omega^{\frac12})$ and $B \in \Psi^{m'}_b(X; {}^b\Omega^{\frac12})$, have Schwartz kernels supported in a small neighbourhood of the corner of X^2. Indeed this can be arranged by appropriately decomposing

$$A = A' + A'', \;\; B = B' + B''$$

where A', and B' have the support property and A'' and B'' have kernels which vanish near the corner. Then A'' and B'' have kernels which are $\mathcal{C}^\infty$ in a neighbourhood of the complete boundary of X^2, $\partial X \times X \cup X \times \partial X$. It follows, as in (4.71), that they in turn are the sums of a pseudodifferential operator, in the usual sense, with kernel supported strictly in the interior of X^2 plus a smooth kernel vanishing to infinite order at the boundary of X^2. The latter maximally residual operator maps $\mathcal{C}^{-\infty}(X; {}^b\Omega^{\frac12})$ into $\dot{\mathcal{C}}^\infty(X; {}^b\Omega^{\frac12})$, and conversely this is a characterization of these operators. From the standard theory of pseudodifferential operators, the product of two b-pseudodifferential operators with kernels supported strictly in the interior of X^2 is of the same type. If either operator is maximally residual then from the regularity properties of pseudodifferential operators so is the product. Thus it follows that $A'' \circ B''$ is the sum of a pseudodifferential operator supported in the interior of X^2 and a maximally residual operator. In fact the same is true for the products $A' \circ B''$ and $A'' \circ B'$ since if B'' or A'' is maximally residual, so is the product (using (4.70)) and if B'' or A'' has kernel supported in the interior then the kernels of A' and B' near the corner are irrelevant, i.e. they can be replaced by kernels which vanish near the corner. Since

$$A \circ B = (A' + A'') \circ (B' + B'') = A' \circ B' + A' \circ B'' + A'' \circ B' + A'' \circ B'' \tag{5.60}$$

[7] Such a proof can be based on the notion of a b-fibration discussed in Exercise 4.7 and the construction of a stretched triple product.

it remains only to consider the product $A' \circ B'$.

Let x be a boundary defining function. A neighbourhood of the boundary of X can be identified with $[0,\epsilon)_x \times \partial X$, and hence a neighbourhood of the corner of X^2 with $[0,\epsilon)_x \times \partial X \times [0,\epsilon)_{x'} \times \partial X$. As discussed in Chapter 4 a neighbourhood of the front face of X^2_b can be identified with the closure of

$$[0,\infty)_s \times [0,\epsilon)_x \times \partial X \times \partial X,\ \ s = x/x'. \tag{5.61}$$

Then the kernel of A is, in view of (4.58), of the form

$$\kappa_A = k(x,s,y,y')|\frac{ds}{s}\frac{dx}{x}dydy'|^{\frac12},$$

where k is singular only at $s=1, y=y'$ and is rapidly vanishing with all derivatives as $s \downarrow 0$ or $s \uparrow \infty$.

The action of the operator can then be written, as in (4.68),

$$A\phi(x,y)|\frac{dx}{x}dy|^{\frac12} = \int_0^\infty \int_{\partial X} k(s,x,y,y')\phi(x/s,y')\frac{ds}{s}dy'|\frac{dx}{x}dy|^{\frac12}. \tag{5.62}$$

Writing ϕ in terms of its Mellin transform gives

$$A\phi(x,y)|\frac{dx}{x}dy|^{\frac12} = \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\partial X} k_M(x,\lambda,y,y')\phi_M(\lambda,y')x^{i\lambda}\frac{ds}{s}dy'|\frac{dx}{x}dy|^{\frac12}, \tag{5.63}$$

where k_M is the Mellin transform of k with respect to the variable s.

Using the variable x' in place of x in (5.61) for the kernel of B gives

$$B\phi(x,y)|\frac{dx}{x}dy|^{\frac12} = \int_0^\infty \int_{\partial X} k'(x/s,s,y,y')\phi(x/s,y')\frac{ds}{s}dy'|\frac{dx}{x}dy|^{\frac12}. \tag{5.64}$$

Taking the Mellin transform this becomes

$$(B\phi)_M(\lambda,y)|dy|^{\frac12} = \int_{\partial X} k'_M(x',\lambda,y,y')(x')^{-i\lambda}\phi(x',y')\frac{dx'}{x'}dy'|dy|^{\frac12}. \tag{5.65}$$

Combining (5.63) and (5.65) expresses the composite operator as

$$AB\phi(x,y)|\frac{dx}{x}dy|^{\frac12} = \int_0^\infty \int_{\partial X} k''(x,x/s,s,y,y')\phi(x/s,y')\frac{ds}{s}dy'|\frac{dx}{x}dy|^{\frac12},$$

where the kernel is
(5.66)
$$k''(x,x',s,y,y') = \frac{1}{2\pi}\int\limits_{\mathbb{R}}\int\limits_{\partial X} k_M(x,\lambda,y,y'')k'_M(x',\lambda,y'',y')s^{i\lambda}\,d\lambda dy''.$$

In this integral x and x' are simply smooth parameters. From the properties of the indicial families, as in (5.14) and the composition formula for pseudodifferential operators on compact manifolds without boundary ([47]), it follows that k'' is indeed the kernel of an element of $\Psi_b^{m+m'}(X;{}^b\Omega^{\frac12})$, proving the proposition.

EXERCISE 5.21. Note that this analysis is directly modeled on one of the standard methods of proof of the composition formula for pseudodifferential operators on $\mathbb{R}^n$, by writing one of the operators in 'left reduced' form and the other in 'right reduced' form. See if you can directly deduce the composition formula from the composition properties of pseudodifferential operators on $\mathbb{R}^n$, by introducing $\log x$ as a variable, and localizing in the tangential variables.

COROLLARY. *The product formula* (5.18) *extends to show that the map* (5.60) *is a homomorphism; hence* (4.101) *also extends to show that* (4.98) *is a homomorphism.*

It is useful to record an extension of the argument used to prove Proposition 5.20; this will be rather helpful in handling remainder terms later.

LEMMA 5.22. *There exists an integer p such that composition of operators defines a bilinear map*

$$\dot{\mathcal{C}}^N(X_b^2;{}^b\Omega^{\frac12})\times\dot{\mathcal{C}}^N(X_b^2;{}^b\Omega^{\frac12})\subset\dot{\mathcal{C}}^{N-p}(X_b^2;{}^b\Omega^{\frac12})\ \forall\ N>p$$
$$\Psi_b^m(X;{}^b\Omega^{\frac12})\times\dot{\mathcal{C}}^N(X_b^2;{}^b\Omega^{\frac12})\subset\dot{\mathcal{C}}^{N-p-m}(X_b^2;{}^b\Omega^{\frac12})\ \forall\ N>p+m.$$

PROOF: See (5.62) - (5.66).

5.10. Polyhomogeneous conormal distributions.

It has been shown above that the inverse of the indicial operator of an elliptic b-differential operator has a kernel which is the sum of a term in the small calculus, plus a $\mathcal{C}^\infty$ term which has a complete asymptotic expansion at the boundaries of $\mathrm{bf}(X_b^2)$. This can be taken as a strong indication of the structure of the kernels of the generalized inverses of elliptic elements of $\mathrm{Diff}_b^m(X;{}^b\Omega^{\frac12})$. The notion of a b-pseudodifferential operator will therefore be expanded accordingly. A parametrix, modulo compact errors, can then be found in this enlarged calculus just using the inverse of the indicial operator. Subsequently it will be shown that a much finer parametrix can be obtained by using iteration, but at some cost in terms of complexity.

Recall that X_b^2 has three distinct pieces of boundary: the front face, $\mathrm{bf}(X_b^2)$, and the left and right boundary faces. The extra terms added to the calculus (for the moment) will be $\mathcal{C}^\infty$ sections of ${}^b\Omega^{\frac{1}{2}}$ up to $\mathrm{bf}(X_b^2)$, but having expansions at lb and rb . These will be polyhomogeneous conormal distributions with respect to the boundary. This notion will be considered in some detail, first with respect to a boundary hypersurface of a compact manifold with boundary, X.

If $\rho \in \mathcal{C}^\infty(X)$ is a boundary defining function, *define*

$$u \in \mathcal{A}^E_{\mathrm{phg}}(X) \Longleftrightarrow u \sim \sum_{(z,k)\in E} \rho^z (\log \rho)^k a_{k,z}. \tag{5.67}$$

Here the coefficients are smooth, $a_{k,z} \in \mathcal{C}^\infty(X)$. The precise meaning of $\sim$ is discussed below.

Consider first the conditions to be imposed on E. Clearly

$$E \subset \mathbb{C} \times \mathbb{N}_0, \quad \mathbb{N}_0 = \{0, 1, \dots\} \tag{5.68}$$

is a sort of 'divisor' (because it will correspond to the singularities of the Mellin transform, see Proposition 5.27). In order for the sum in (5.67) to be reasonably sensible it is surely necessary that

$$E \text{ is discrete.} \tag{5.69}$$

Furthermore all but a finite number of terms should vanish to any fixed order at $\rho = 0$. Thus it is natural to require

$$(z_j, k_j) \in E, \ |(z_j, k_j)| \longrightarrow \infty \implies \operatorname{Re} z_j \longrightarrow \infty. \tag{5.70}$$

Multiplying ρ by a positive $\mathcal{C}^\infty$ function clearly produces from any power of $\log \rho$ the lower powers of $\log \rho$. Since the space is supposed to be independent of the choice of ρ it is also natural to require

$$(z, k) \in E \Longrightarrow (z, l) \in E, \ l \in \mathbb{N}_0, \ 0 \le l \le k. \tag{5.71}$$

Generally the space is to be a $\mathcal{C}^\infty(X)$-module, i.e. to be preserved under multiplication by $\mathcal{C}^\infty(X)$, and this will follow from:

$$(z, k) \in E \Longrightarrow (z + j, k) \in E, \ j \in \mathbb{N}_0. \tag{5.72}$$

DEFINITION 5.23. An index set (or sometimes a $\mathcal{C}^\infty$ index set for emphasis) is a subset (5.68) which satisfies (5.69), (5.70), (5.71) and (5.72).

The last condition, (5.72) appears because the coefficients in (5.67) are $\mathcal{C}^\infty$ functions. They therefore have Taylor expansions in powers of ρ. Thus (5.72) might as well be imposed since it corresponds to the 'lower order terms' in such an expansion. The discussion of the model problem above takes place on a space (the normal bundle) with an $\mathbb{R}^+$-action. Then it makes sense to choose a homogeneous defining function and demand that the coefficients in (5.67) be homogeneous of degree zero. In that case it is not necessary to impose the condition that the space be a $\mathcal{C}^\infty(X)$-module, and then (5.72) will be dropped. A set satisfying all the conditions except this last one will be called an *absolute* index set. As will become clear below, $\mathcal{C}^\infty$ index sets often arise simply by adding exponents to an absolute index set to get (5.72). In doing so rather unpleasant complications arise with (5.71). This is the conundrum of accidental multiplicities.

If E is an index set then (5.67) is to be interpreted as requiring

$$\begin{aligned} &u \in \mathcal{A}^E_{\text{phg}}(X) \text{ iff } \exists\ a_{z,k} \in \mathcal{C}^\infty(X)\ \forall\ (z,k) \in E \\ &\text{s.t. } \forall\ N,\ u - \sum_{\substack{(z,k)\in E \\ \operatorname{Re} z \le N}} \rho^z (\log\rho)^k a_{z,k} \in \dot{\mathcal{C}}^N(X), \end{aligned} \tag{5.73}$$

where $\dot{\mathcal{C}}^N(X)$ is the space of functions which are N times differentiable on X and vanish at ∂X with all derivatives up to order N. The condition (5.70) means that the sum in (5.73) is always finite. Clearly $\mathcal{C}^\infty(X)\cdot\dot{\mathcal{C}}^N(X) = \dot{\mathcal{C}}^N(X)$ so indeed

$$\mathcal{C}^\infty(X)\cdot\mathcal{A}^E_{\text{phg}}(X) = \mathcal{A}^E_{\text{phg}}(X),$$

for any index set because of (5.72). Moreover the same definition can be used for the space $\mathcal{A}^E_{\text{phg}}(X;F)$, where F is any vector bundle F over X, just replacing $a_{z,k} \in \mathcal{C}^\infty(X)$ with $a_{z,k} \in \mathcal{C}^\infty(X;F)$.

It is reassuring to know that there are many functions with non-trivial expansions and this is shown by a lemma of Borel which in present circumstances takes the form:

LEMMA 5.24. *Let X be a compact manifold with boundary, let E be an index set and suppose $a_{z,k} \in \mathcal{C}^\infty(X)$ is given for each $(z,k) \in E$ then there exists $u \in \mathcal{A}^E_{\text{phg}}(X)$ satisfying (5.67) and if $u' \in \mathcal{A}^E_{\text{phg}}(X)$ is any other element with the same expansion then $u' - u \in \dot{\mathcal{C}}^\infty(X)$.*

PROOF: Choose $\mu \in \mathcal{C}^\infty_c([0,\infty))$ with $\mu(x) = 1$ on $[0,1]$. It is straightforward to show that if $\epsilon_{z,k} \in (0,1)$ are positive constants then for each $N \in \mathbb{N}$ the series

$$\sum_{\substack{(z,k)\in E \\ \operatorname{Re} z > N}} \mu\Big(\frac{\rho}{\epsilon_{z,k}}\Big)\rho^z(\log\rho)^k a_{z,k} \text{ converges absolutely in } \dot{\mathcal{C}}^N(X) \tag{5.74}$$

provided the $\epsilon_{z,k}$ decrease fast enough. More precisely there is, for each N, a sequence $\epsilon^{(N)}_{z,k}$ such that (5.74) holds for that N provided

$$\epsilon_{z,k} < \epsilon^{(N)}_{z,k} \ \forall \ \operatorname{Re} z > N.$$

Since $\operatorname{Re} z < N$ eventually because of (5.70), all these conditions taken together still reduce to only a finite number of conditions on each $\epsilon_{z,k}$. Therefore these constants can be chosen in such a way that (5.74) holds for all N. Then the series converges,

$$u = \sum_{(z,k)\in E} \mu(\frac{\rho}{\epsilon_{z,k}})\rho^z(\log\rho)^k a_{z,k} \in \mathcal{A}^E_{\text{phg}}(X),$$

to an element satisfying (5.67).

The essential uniqueness of this asymptotic sum follows directly from the definition and the fact that

$$\dot{\mathcal{C}}^\infty(X) = \bigcap_N \dot{\mathcal{C}}^N(X).$$

These polyhomogeneous conormal distributions are intimately associated to the b-pseudodifferential operators, as will be seen below (see Proposition 5.59) since each element of the null space of a b-elliptic operator is polyhomogeneous. For the moment consider the more prosaic fact that the b-pseudodifferential operators act on the polyhomogeneous conormal distributions:

PROPOSITION 5.25. *If X is a compact manifold with boundary then for any index set E each $A \in \Psi^*_b(X;{}^b\Omega^{\frac12})$ defines an operator*

$$A\colon \mathcal{A}^E_{\text{phg}}(X;{}^b\Omega^{\frac12}) \longrightarrow \mathcal{A}^E_{\text{phg}}(X;{}^b\Omega^{\frac12}).$$

PROOF: The proof of (4.70) in Proposition 4.29 actually shows that

$$A\colon \dot{\mathcal{C}}^N(X;{}^b\Omega^{\frac12}) \longrightarrow \dot{\mathcal{C}}^{N'}(X;{}^b\Omega^{\frac12}) \text{ where } N' \longrightarrow \infty \text{ as } N \longrightarrow \infty.$$

Thus it suffices to show that the structure of the finite sum in (5.73) is preserved, i.e.

$$\text{(5.75)} \qquad \begin{gathered} A\left(x^z\sum_{k\le l}(\log x)^l a_k\right) = x^z\sum_{k\le l}(\log x)^l b_l, \\ a_l \in \mathcal{C}^\infty(X;{}^b\Omega^{\frac12}) \Longrightarrow b_l \in \mathcal{C}^\infty(X;{}^b\Omega^{\frac12}). \end{gathered}$$

In case $l = 0$ (5.75) just reduces to

$$x^{-z}Ax^{z} : \mathcal{C}^\infty(X; {}^b\Omega^{\frac12}) \longrightarrow \mathcal{C}^\infty(X; {}^b\Omega^{\frac12}) \tag{5.76}$$

and so follows from the invariance of $\Psi_b^m(X; {}^b\Omega^{\frac12})$, for each m, under conjugation by complex powers of x, as shown in Proposition 5.7 and from the fact that the elements of the small calculus preserve $\mathcal{C}^\infty$ regularity, shown in Proposition 4.34. The general case follows by differentiating (5.76) with respect to z.

If X is a manifold with corners various extensions of $\mathcal{A}^E_{\text{phg}}(X)$ can be defined, although fortunately only particular cases occur here (see [63] for the general case). Let $H_1, \dots, H_k$ be the boundary hypersurfaces of X. By an *index family* for X is meant $\mathcal{E} = (E_1, \dots E_k)$, where each E_j is an index set associated to H_j. If all but one of the E_j are the special index set $E = \{(l, 0); l \in \mathbb{N}_0\}$, which will simply be written '0' and corresponds to

$$\mathcal{A}^E_{\text{phg}}(X) = \mathcal{A}^0_{\text{phg}}(X) = \mathcal{C}^\infty(X),$$

then the definition (5.73) can still be used, with ρ replaced by ρ_p, corresponding to the one non-trivial index set:

$$\begin{gathered}\mathcal{E} = (E_1, \dots E_k), E_j = 0, j \neq p, \text{ then } u \in \mathcal{A}^{\mathcal{E}}_{\text{phg}}(X) \Longleftrightarrow \\ \exists\, a_{z,k} \in \mathcal{C}^\infty(X)\ \forall\ (z,k) \in E_p \text{ s.t. } u \sim \sum_{(z,k)\in E_p} \rho_p^z (\log \rho_p)^k a_{z,k}.\end{gathered} \tag{5.77}$$

In the definition, (5.73), of $\sim$ the error term should be taken as the space of functions which are N times differentiable and vanish to order N at H_p. More generally the definition can be extended to the case of an index family $\mathcal{E} = (E_1, \dots, E_k)$ with the property that the boundary hypersurfaces which correspond to index sets other than 0 are all disjoint. That is,

$$\begin{gathered}\mathcal{E} = (E_1, \dots, E_k) \text{ where } E_j = 0 \text{ for } j \notin I \subset \{1, \dots, k\}, \text{ and} \\ j, k \in I,\ j \neq k \Longrightarrow H_j \cap H_k = \emptyset.\end{gathered} \tag{5.78}$$

The expansions at the 'non-trivial' boundary hypersurfaces are then independent of one another. Choose a partition of unity

$$\begin{gathered}\phi_j \in \mathcal{C}^\infty(X),\ j \in I,\ \sum_{j\in I} \phi_j = 1 \quad \text{on } X \\ \operatorname{supp}(\phi_j) \cap H_p = \emptyset \quad \text{if } j, p \in I,\ j \neq p.\end{gathered} \tag{5.79}$$

Then set

$$\mathcal{E}_j = (E_1^{(j)}, \dots, E_k^{(j)}),\ E_l^{(j)} = \begin{cases} 0 & l \neq j, \\ E_j & l = j. \end{cases} \quad \text{for each } j \in I.$$

This is an index family of the type (5.77) for each $j \in I$, so the previous definition applies and

$$u \in \mathcal{A}^{\mathcal{E}}_{\text{phg}}(X;F) \Longleftrightarrow \phi_j u \in \mathcal{A}^{\mathcal{E}_j}_{\text{phg}}(X;F)\ \forall\ j \in I.$$

This is clearly independent of the choice of partition of unity satisfying (5.79) since for another such partition, ϕ'_j, $(\phi'_j - \phi_j)u$ is always $\mathcal{C}^\infty$ and supported away from the boundary hypersurfaces H_j, $j \in I$. Later a case where the index sets on two boundary hypersurfaces which meet are both non-trivial will have to be considered.

If $\mathcal{E}$ is an index family as in (5.78) for a manifold with corners, X, let $Y = H_j$ be a boundary hypersurface with $j \notin I$, i.e. the index set for $Y = H_j$ is 0. Then each element of $\mathcal{A}^{\mathcal{E}}_{\text{phg}}(X;F)$ is smooth up to Y. Now Y is also a manifold with corners where each boundary hypersurface is a component of the intersection of a boundary hypersurface of X with Y. Let $\mathcal{F}$ be the index family for Y, which associates to each boundary hypersurface the index set of the boundary hypersurface of X from which it arises.

Lemma 5.26. *If $\mathcal{E}$ is an index family satisfying (5.78) for X and $Y = H_j$, for some $j \notin I$, is a boundary hypersurface of X with index set 0 then with $\mathcal{F}$ defined as above,*

$$\mathcal{A}^{\mathcal{E}}_{\text{phg}}(X;F) \ni u \longmapsto u_{\restriction Y} \in \mathcal{A}^{\mathcal{F}}_{\text{phg}}(Y;F) \tag{5.80}$$

is surjective.

Proof: The expansion for $u_{\restriction Y}$ required for (5.80) follows immediately from that for u. To see the surjectivity first extend a given element $v \in \mathcal{A}^{\mathcal{F}}_{\text{phg}}(Y;F)$ off Y into X to be independent of a normal variable near Y. Multiplying by $\phi \in \mathcal{C}^\infty(X)$ which is 1 near Y and has support sufficiently close to Y clearly gives an element of $\mathcal{A}^{\mathcal{E}}_{\text{phg}}(X;F)$ which restricts to Y to give v.

5.11. Mellin transform and polyhomogeneity.

There is an intimate connection between the Mellin transform and polyhomogeneity at a boundary. In Lemma 5.16 the Mellin transform of elements of weighted Sobolev spaces on $\widetilde{X}$ was considered. By choosing a trivialization of the normal bundle these are just results about the Mellin transform of functions on $[-1,1]_r \times \partial X$ in which ∂X is really a space of parameters. A useful characterization of polyhomogeneous conormal functions is easily obtained. It has really already been used in the proof of Lemma 5.14 and has therefore guided the definition of polyhomogeneity, so should not come as a surprise:

PROPOSITION 5.27. *Let Y be a compact manifold with corners and let $\mathcal{E} = (E, \emptyset, 0, \ldots, 0)$ be the index family for $[-1,1] \times Y$ which assigns index set E to $\{-1\} \times Y$, the trivial index set $\emptyset$ to $\{1\} \times Y$ and the $\mathcal{C}^\infty$ index set to all boundary faces of Y, then the Mellin transform*

$$u_M(\lambda, y) = \int_0^\infty x^{-i\lambda} u(\frac{x-1}{x+1}, y) \frac{dx}{x}$$

gives an isomorphism from $\mathcal{A}^{\mathcal{E}}_{\mathrm{phg}}([-1,1] \times Y)$ to the space of meromorphic functions with values in $\mathcal{C}^\infty(Y)$ having poles of order k only at points $\lambda = -iz \in \mathbb{C}$ such that $(z, k-1) \in E$ and satisfying for each large N

$$\|u_M(\lambda, \cdot)\|_N \le C_N(1 + |\lambda|)^{-N} \text{ in } |\operatorname{Im}\lambda| \le N, |\operatorname{Re}\lambda| \ge C_N, \tag{5.81}$$

where $\|\cdot\|_N$ is a norm on $\mathcal{C}^N(Y)$.

The estimates (5.81) are consistent with the meromorphy of u_M since there are only finitely many poles in any strip $\operatorname{Im}\lambda \ge -N$.

PROOF: If $u \in \mathcal{C}^N([-1,1] \times Y)$ vanishes with its first N derivatives at $\{-1\} \times Y$ and $\{1\} \times Y$ then the integral defining u_M converges absolutely for $|\operatorname{Im} z| < N$ and, using the identities as in (5.5), satisfies (5.81) with N replaced by $N-1$. In particular the desired estimates hold if $E = \emptyset$ with u_M entire. It can therefore be assumed that u has support near $\{-1\} \times Y$ and the variable x can be used, so that $[-1,1] \times Y$ is replaced by $[0, \infty) \times Y$, with u supported in $x < 1$. The remainder terms in the expansion (5.77) become arbitrarily smooth and vanish with their derivatives at $x = 0$, so to prove (5.81) it suffices to check the estimates, and the meromorphy, for each term in the expansion.

Thus it is enough to suppose that u is of the form

$$u = x^z (\log x)^k \phi(x) \psi(y), \;\; \psi \in \mathcal{C}^\infty(Y),$$

where $\phi \in \mathcal{C}^\infty_c([0, \infty))$ takes the value 1 in $x \le \frac{1}{2}$. Then the Mellin transform of u is just

$$u_M(\lambda, y) = i^k \left(\frac{\partial}{\partial\lambda}\right)^k v_M(\lambda + iz, y), \;\; v = \phi(x)\psi(y).$$

Thus it is enough to show that v_M has only a simple pole at $\lambda = 0$ and satisfies (5.81). The meromorphy property follows by writing the Mellin transform as

$$v_M(\lambda, y) = \psi(y)[\frac{i}{\lambda} 2^{-i\lambda} + \int_{\frac{1}{2}}^1 x^{-i\lambda} \phi(x) \frac{dx}{x}]$$

and noting that the second term is entire. The estimates (5.81) follow from integration by parts, since

$$\lambda v_M(\lambda) = w_M(\lambda),\ \ w(x) = -ix\frac{\partial}{\partial x}\phi(x)\psi(y).$$

This shows that the range of the Mellin transform has the properties indicated. Moreover u can be recovered from its Mellin transform by (5.3) for $r >> 0$. Thus it suffices to show that u, given by this formula from u_M with the meromorphy property and satisfying (5.81) is, in $\mathcal{A}^{\mathcal{E}}_{\text{phg}}([-1,1]\times Y)$. This is essentially a repetition of the proof of Lemma 5.14, with the terms in the expansion arising from Cauchy's formula, so the details are omitted.

5.12. Boundary terms.

With this preamble additional boundary terms can now be added to the calculus. Let E_{lb} and E_{rb} be index sets associated to $\text{lb}(X^2_b)$ and $\text{rb}(X^2_b)$ and set $\mathcal{E}' = (E_{\text{lb}}, E_{\text{rb}}, 0)$, corresponding to the ordering lb, rb, bf of the boundary hypersurfaces of X^2_b. Then define

$$\widetilde{\Psi}^{-\infty,\mathcal{E}}_b(X;{}^b\Omega^{\frac12}) \overset{\text{def}}{=} \mathcal{A}^{\mathcal{E}'}_{\text{phg}}(X^2_b;{}^b\Omega^{\frac12}). \tag{5.82}$$

In this notation, $\mathcal{E} = (E_{\text{lb}}, E_{\text{rb}})$ is thought of as an index family for X^2, $\text{lb} = \partial X \times X$, $\text{rb} = X \times \partial X$. This is only part of the full calculus; there is still just a little bit more to come. A parametrix for an elliptic differential operator will be found in the space

$$\widetilde{\Psi}^{m,\mathcal{E}}_{b,\text{os}}(X;{}^b\Omega^{\frac12}) \overset{\text{def}}{=} \Psi^m_{b,\text{os}}(X;{}^b\Omega^{\frac12}) + \widetilde{\Psi}^{-\infty,\mathcal{E}}_b(X;{}^b\Omega^{\frac12}). \tag{5.83}$$

Notice that an element $A \in \widetilde{\Psi}^{m,\mathcal{E}}_{b,\text{os}}(X;{}^b\Omega^{\frac12})$, or rather its kernel, can be characterized by taking a partition of unity

$$\begin{gathered}\phi_{\text{lb}}, \phi_{\text{rb}}, \phi_\Delta \in \mathcal{C}^\infty(X^2_b),\ \ \phi_{\text{lb}} + \phi_{\text{rb}} + \phi_\Delta = 1\\ \text{supp}(\phi_{\text{lb}}) \cap [\text{rb} \sqcup \Delta_b] = \emptyset\\ \text{supp}(\phi_{\text{rb}}) \cap [\text{lb} \sqcup \Delta_b] = \emptyset\\ \text{supp}(\phi_\Delta) \cap [\text{lb} \sqcup \text{rb}] = \emptyset\end{gathered}$$

and then

$$\begin{gathered}A \in \widetilde{\Psi}^{m,\mathcal{E}}_b(X;{}^b\Omega^{\frac12}) \Longleftrightarrow\\ \phi_\Delta A \in \Psi^m_b(X;{}^b\Omega^{\frac12}),\\ \phi_{\text{lb}} A \in \mathcal{A}^{(E_{\text{lb}},0,0)}_{\text{phg}}(X^2_b;{}^b\Omega^{\frac12}),\\ \phi_{\text{rb}} A \in \mathcal{A}^{(0,E_{\text{rb}},0)}_{\text{phg}}(X^2_b;{}^b\Omega^{\frac12}).\end{gathered}$$

From this it follows that the intersection of the small calculus and the extra boundary calculus is just the residual space for the diagonal symbol:

$$\Psi^m_b(X;{}^b\Omega^{\frac12}) \cap \widetilde{\Psi}^{-\infty,\mathcal{E}}_b(X;{}^b\Omega^{\frac12}) = \Psi^{-\infty}_b(X;{}^b\Omega^{\frac12}) \quad \forall\ m, \mathcal{E}.$$

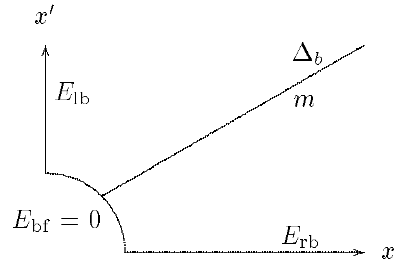

Figure 10. Singularities of $A \in \widetilde{\Psi}_b^{m,\mathcal{E}}(X; {}^b\Omega^{\frac{1}{2}})$.

5.13. True parametrix.

To get started on the parametrix construction recall what was shown above about the inverse Mellin transform of the inverse of the indicial family of an elliptic operator. Restating Lemma 5.16 in terms of the new notation for polyhomogeneous conormal distributions

(5.84)
$$\begin{gathered} Q \in \mathrm{Diff}^k_{b,I}(\widetilde{X}; {}^b\Omega^{\frac{1}{2}}) \text{ elliptic, } \alpha \notin -\operatorname{Im}\operatorname{spec}_b(Q) \Longrightarrow \\ K(s, \cdot) = \frac{1}{2\pi} \int\limits_{\operatorname{Im}\lambda = -\alpha} s^{i\lambda} \widehat{Q}(\lambda)^{-1} d\lambda = A + B,\ B \in \mathcal{A}^{\mathcal{E}(\alpha)}_{\mathrm{phg}}(\mathrm{bf}(X^2_b); {}^b\Omega^{\frac{1}{2}}). \end{gathered}$$

Here A is just the 'crude' parametrix discussed in Chapter 4. The front face $\mathrm{bf}(X^2_b)$ has a natural $\mathbb{R}^+$-action and the index family

$$\mathcal{E}(\alpha) = (E_{\mathrm{lb}}, E_{\mathrm{rb}}) = (E^+(\alpha), E^-(\alpha))$$

for bf correspondingly consists of two absolute index sets:

(5.85)
$$\begin{aligned} E^{\pm}(\alpha) = \big\{(z, k); \widehat{Q}(\lambda)^{-1} &\text{ has a pole at } \lambda = \mp iz \\ &\text{of order at most } k+1 \text{ and } \pm\operatorname{Im}\lambda < \mp\alpha\big\}. \end{aligned}$$

Recall that the *order* of a pole is the least integer l such that $(\lambda + iz)^l \widehat{Q}(\lambda)^{-1}$ is regular near $\lambda = -iz$. Using this consider the smallest $\mathcal{C}^\infty$ index sets containing $E^{\pm}(\alpha)$:

$$(5.86) \quad \widetilde{E}^{\pm}(\alpha) = \big\{(z, k) \in \mathbb{C} \times \mathbb{N}_0; (z - r, k) \in E^{\pm}(\alpha) \text{ for some } r \in \mathbb{N}_0\big\}.$$

A little more notation is also useful for the spaces of b-pseudodifferential operators. Namely let

$$\rho_{\mathrm{bf}}\widetilde{\Psi}_b^{-\infty,\mathcal{E}}(X;{}^b\Omega^{\frac{1}{2}}) \subset \widetilde{\Psi}_b^{-\infty,\mathcal{E}}(X;{}^b\Omega^{\frac{1}{2}}),$$

with a formal coefficient ρ_{bf}, stand for the subspace of operators with kernel of the form $\rho_{\mathrm{bf}}B$, where B is in $\widetilde{\Psi}_b^{-\infty,\mathcal{E}}(X;{}^b\Omega^{\frac{1}{2}})$.

PROPOSITION 5.28. *If $P \in \mathrm{Diff}_b^k(X;{}^b\Omega^{\frac{1}{2}})$ is elliptic then for each $\alpha \in \mathbb{R}$ such that $\alpha \notin -\operatorname{Im}\operatorname{spec}_b(P)$ there exists*

$$G_\alpha \in \widetilde{\Psi}_b^{-k,\widetilde{\mathcal{E}}(\alpha)}(X;{}^b\Omega^{\frac{1}{2}}), \tag{5.87}$$

where $\widetilde{\mathcal{E}}(\alpha) = (\widetilde{E}^+(\alpha), \widetilde{E}^-(\alpha))$ consists of the index sets given by (5.86) and

$$P \circ G_\alpha = \mathrm{Id} - R_\alpha,\ \ R_\alpha \in \rho_{\mathrm{bf}}\widetilde{\Psi}_b^{-\infty,\widetilde{\mathcal{E}}(\alpha)}(X;{}^b\Omega^{\frac{1}{2}}). \tag{5.88}$$

Clearly it is necessary to analyze $P \circ G_\alpha$ where G_α is as in (5.87). Using the decomposition (5.83) and (4.80), only the action on the new part of the calculus is needed:

LEMMA 5.29. *For any index family $\mathcal{E} = (E_{\mathrm{lb}}, E_{\mathrm{rb}})$ and $P \in \mathrm{Diff}_b^k(X;{}^b\Omega^{\frac{1}{2}})$ composition gives*

$$\widetilde{\Psi}_b^{-\infty,\mathcal{E}}(X;{}^b\Omega^{\frac{1}{2}}) \ni G \longmapsto P \circ G \in \widetilde{\Psi}_b^{-\infty,\mathcal{E}}(X;{}^b\Omega^{\frac{1}{2}}). \tag{5.89}$$

PROOF: Since $\mathrm{Diff}_b^k(X)$ is generated by $\mathcal{V}_b(X)$ which lifts from the left factor of X^2 into $\mathcal{V}_b(X_b^2)$ (Lemma 4.3) it suffices to show that

$$\mathcal{V}_b(X_b^2) \cdot \mathcal{A}_{\mathrm{phg}}^{\mathcal{E}}(X_b^2) \subset \mathcal{A}_{\mathrm{phg}}^{\mathcal{E}}(X_b^2)$$

which is rather obvious, since it amounts to the observation that if $k \geq 1$

$$x\frac{\partial}{\partial x}\left(x^z(\log x)^k\right) = zx^z(\log x)^k + kx^z(\log x)^{k-1}$$

and of course $x\partial x^z/\partial x = z\ x^z$. This, by the way, is a good reason for having the condition (5.71) on index sets.

Not only does this argument prove the lemma but it gives a little more. Namely just as in (4.89) the indicial homomorphism can be defined for the boundary terms by

$$I : \widetilde{\Psi}_b^{-\infty,\mathcal{E}}(X;{}^b\Omega^{\frac{1}{2}}) \ni B \longmapsto B_{|\mathrm{bf}(X_b^2)} \in \mathcal{A}_{\mathrm{phg}}^{\mathcal{E}}(\mathrm{bf}(X_b^2);{}^b\Omega^{\frac{1}{2}}), \tag{5.90}$$

where on the right $\mathcal{E}$ is interpreted as an index family for $\mathrm{bf}(X_b^2)$. Then

$$I(P \circ B) = I(P) \circ I(B),\ \ P \in \mathrm{Diff}_b^k(X; {}^b\Omega^{\frac12}),\ \ B \in \widetilde{\Psi}_b^{-\infty,\mathcal{E}}(X; {}^b\Omega^{\frac12}).$$

Here the indicial operator of P is acting as a b-differential operator on the front face of X_b^2. This is enough preliminary orientation to begin the

PROOF OF PROPOSITION 5.28: The subspace

$$\rho_{\mathrm{bf}}\widetilde{\Psi}_b^{-\infty,\widetilde{\mathcal{E}}(\alpha)}(X; {}^b\Omega^{\frac12}) \subset \widetilde{\Psi}_b^{-\infty,\widetilde{\mathcal{E}}(\alpha)}(X; {}^b\Omega^{\frac12})$$

is just the null space of the indicial homomorphism (5.90). Indeed the extension result in Lemma 5.24 shows that

$$\begin{aligned}0 \longrightarrow \rho_{\mathrm{bf}}\widetilde{\Psi}_b^{-\infty,\mathcal{E}}(X; {}^b\Omega^{\frac12}) &\hookrightarrow \widetilde{\Psi}_b^{-\infty,\mathcal{E}}(X; {}^b\Omega^{\frac12}) \\ &\xrightarrow{I} \mathcal{A}_{\mathrm{phg}}^{\mathcal{E}}(\mathrm{bf}(X_b^2); {}^b\Omega^{\frac12}) \longrightarrow 0\end{aligned} \tag{5.91}$$

is exact. If $A \in \Psi_b^{-k}(X; {}^b\Omega^{\frac12})$ is the 'small' parametrix of Proposition 4.38 then $G = A + G'$ satisfies (5.88) provided

$$I(P) \circ I(G') = \mathrm{Id} - I(P) \circ I(A).$$

That this has a solution of the desired type is the content of (5.84), i.e. it is only necessary to take $I(G') = B$, which is possible because of (5.91).

Lemma 5.29 can be extended to the case that P itself is a b-pseudodifferential operator but to do so requires a further composition formula, at least extending (5.89) to the case $P \in \Psi_b^m(X; {}^b\Omega^{\frac12})$. This is done in Proposition 5.46 below.

5.14. Finitely residual terms.

In (5.42) the weighted b-Sobolev spaces were defined for a manifold with boundary. The definition uses the Lie algebra of smooth vector fields, $\mathcal{V}_b(X)$, in the form of its (filtered) enveloping algebra $\mathrm{Diff}_b^m(X)$, and the L^2 space of b-half densities. The elements are well defined on a manifolds with corners, such as X^2 and X_b^2. Replacing X by either of these spaces in (5.42) defines the weighted Sobolev spaces, at least of positive order. The blow-down map gives an isomorphism

$$\beta_b^* : L^2(X^2; {}^b\Omega^{\frac12}) \longleftrightarrow L^2(X_b^2; {}^b\Omega^{\frac12}) \tag{5.92}$$

because of (4.55) and this extends to all regularity orders.

LEMMA 5.30. *The blow-down map* (4.1) *gives an isomorphism*

$$\beta_b^*: H_b^m(X^2;{}^b\Omega^{\frac12}) \longleftrightarrow H_b^m(X_b^2;{}^b\Omega^{\frac12}),\ \forall\ m\in\mathbb{N}. \tag{5.93}$$

PROOF: This follows directly from (5.92) and Lemma 4.10.

EXERCISE 5.31. Show that (5.93) extends to an isomorphism for all $m\in\mathbb{Z}$.

Since X^2 has two boundary hypersurfaces (assuming that the boundary is connected) and X_b^2 has three, there is a corresponding variety of weighted spaces:

$$\begin{gathered}\rho_{\rm lb}^{\alpha}\rho_{\rm rb}^{\beta}H_b^m(X^2;{}^b\Omega^{\frac12}) = \\ \left\{u\in\mathcal{C}^{-\infty}(X^2;{}^b\Omega^{\frac12});\, u=\rho_{\rm lb}^{\alpha}\rho_{\rm rb}^{\beta}v,\, v\in H_b^m(X^2;{}^b\Omega^{\frac12})\right\}\end{gathered} \tag{5.94}$$

$$\begin{gathered}\rho_{\rm lb}^{\alpha}\rho_{\rm rb}^{\beta}\rho_{\rm bf}^{\gamma}H_b^m(X_b^2;{}^b\Omega^{\frac12}) = \\ \left\{u\in\mathcal{C}^{-\infty}(X_b^2;{}^b\Omega^{\frac12});\, u=\rho_{\rm lb}^{\alpha}\rho_{\rm rb}^{\beta}\rho_{\rm bf}^{\gamma}v,\, v\in H_b^m(X^2;{}^b\Omega^{\frac12})\right\},\end{gathered} \tag{5.95}$$

where α,β and c are real constants. Under the blow-down map

$$\beta_b^*\rho_{\rm lb}=\rho_{\rm bf}\rho_{\rm lb},\ \beta_b^*\rho_{\rm rb}=\rho_{\rm bf}\rho_{\rm rb},$$

so from (5.93), (5.94) and (5.95) it follows that

$$\beta_b^*: \rho_{\rm lb}^{\alpha}\rho_{\rm rb}^{\beta}H_b^m(X^2;{}^b\Omega^{\frac12}) \longleftrightarrow \rho_{\rm lb}^{\alpha}\rho_{\rm rb}^{\beta}\rho_{\rm bf}^{\alpha+\beta}H_b^m(X_b^2;{}^b\Omega^{\frac12}). \tag{5.96}$$

The elements of $\rho_{\rm lb}^{\alpha}\rho_{\rm rb}^{\beta}H_b^m(X^2;{}^b\Omega^{\frac12})$ can be considered as operators. In particular the remainder term in (5.88) is of this type. For an index set let

$$\inf E=\min\{\operatorname{Re}z;(z,0)\in E\} \tag{5.97}$$

measure the smallest power which can occur, then:

LEMMA 5.32. *For any index family* $\mathcal{E}=(E_{\rm lb},E_{\rm rb})$ *for* X^2

$$\begin{gathered}\rho_{\rm bf}\widetilde{\Psi}_b^{-\infty,\mathcal{E}}(X;{}^b\Omega^{\frac12})\subset\rho_{\rm lb}^{\alpha}\rho_{\rm rb}^{\beta}H_b^{\infty}(X^2;{}^b\Omega^{\frac12}) \text{ provided} \\ \alpha<\inf E_{\rm lb},\ \beta<\inf E_{\rm rb} \text{ and } \alpha+\beta<1.\end{gathered} \tag{5.98}$$

PROOF: If A is the kernel of an element of the space on the left in (5.98) then

$$\begin{gathered}\operatorname{Diff}_b^*(X_b^2;{}^b\Omega^{\frac12})A\subset\rho_{\rm lb}^{\alpha}\rho_{\rm rb}^{\beta}\rho_{\rm bf}L^{\infty}(X_b^2;{}^b\Omega^{\frac12}) \text{ provided} \\ \alpha<\inf E_{\rm lb},\beta<\inf E_{\rm rb}.\end{gathered}$$

For any $\epsilon>0$

$$\rho_{\rm lb}^{\epsilon}\rho_{\rm rb}^{\epsilon}\rho_{\rm bf}^{\epsilon}L^{\infty}(X_b^2;{}^b\Omega^{\frac12})\subset L^2(X_b^2;{}^b\Omega^{\frac12}),$$

so $A\in\rho_{\rm lb}^{\alpha-\epsilon}\rho_{\rm rb}^{\beta-\epsilon}\rho_{\rm bf}^{1-\epsilon}\in H_b^{\infty}(X_b^2;{}^b\Omega^{\frac12})$ and then (5.98) follows from (5.96).

By definition the space $L^2(X^2;{}^b\Omega^{\frac12})$ consists of the Hilbert-Schmidt operators on $L^2(X;{}^b\Omega^{\frac12})$. Conjugating by powers of ρ shows that

Lemma 5.33. *The space $\rho_{\mathrm{lb}}^{\alpha}\rho_{\mathrm{rb}}^{\beta}H_b^{\infty}(X^2;{}^b\Omega^{\frac12})$ consists of Hilbert-Schmidt operators on $\rho^{\alpha}L^2(X;{}^b\Omega^{\frac12})$, provided $\beta \ge -\alpha$*

In particular:

Corollary. *In Proposition 5.28 the remainder term for the parametrix satisfies*

$$R_{\alpha} \in \rho_{\mathrm{lb}}^{\alpha}\rho_{\mathrm{rb}}^{\beta}H_b^{\infty}(X^2;{}^b\Omega^{\frac12}) \text{ provided} \\ \alpha < \inf E^{+}(\alpha),\ \beta < \inf E^{-}(\alpha),\ \alpha+\beta<1 \tag{5.99}$$

and is therefore Hilbert-Schmidt on $\rho^{\alpha}L^2(X;{}^b\Omega^{\frac12})$.

5.15. Boundedness on Sobolev spaces.

To use the parametrix effectively it is important to know its boundedness properties, particularly on the b-Sobolev spaces.

Theorem 5.34. *Any $A \in \widetilde{\Psi}_b^{m,\mathcal{E}}(X;{}^b\Omega^{\frac12})$, $m \in \mathbb{Z}$, is bounded as an operator*

$$A\colon \rho^{\alpha}H_b^{M}(X;{}^b\Omega^{\frac12}) \longrightarrow \rho^{\beta}H_b^{M-m}(X;{}^b\Omega^{\frac12}),\ M \in \mathbb{Z}, \\ \text{provided } \beta \le \alpha,\ \alpha + \inf E_{\mathrm{rb}} > 0,\ \inf E_{\mathrm{lb}} > \beta. \tag{5.100}$$

Proof: The operator can be divided into a part in the small calculus and a part of order $-\infty$. For the part in the small calculus the elegant symbolic argument of Hörmander ([46]) can be used. Since the small calculus is invariant under conjugation by complex powers of defining functions, it suffices to take $\alpha=\beta=0$ in this case and show that

$$A \in \Psi_b^{m}(X;{}^b\Omega^{\frac12}) \Longrightarrow A\colon H_b^{M}(X;{}^b\Omega^{\frac12}) \longrightarrow H_b^{M-m}(X;{}^b\Omega^{\frac12}). \tag{5.101}$$

Hörmander's argument reduces (5.101) to the case $m=-\infty$ which will be assumed for the moment, since it has to be checked separately anyway. Since parametrices have been obtained for elliptic operators it is actually enough to prove (5.101) for $M=m=0$. Then a standard symbolic argument (see [46]) allows the extraction of an approximate square root, in the sense that

$$-A^*A + C = B^*B - R,\ B \in \Psi_b^{0}(X;{}^b\Omega^{\frac12}),\ R \in \Psi_b^{-\infty}(X;{}^b\Omega^{\frac12}), \tag{5.102}$$

provided $C>0$ is large enough that the symbol of $-A^*A+C$ is strictly positive near infinity (see Exercise 5.35). From (5.102) it follows that

$$\langle Au, Au\rangle \le C\|u\|^2 + \langle Ru, u\rangle.$$

Thus (5.100) needs only to be checked when $m = -\infty$.

Using the general composition result it is enough to show that

$$|\langle Ru, u\rangle| \le C\|u\|^2, u \in \dot{\mathcal{C}}^\infty(X; {}^b\Omega^{\frac{1}{2}}), \tag{5.103}$$

for these operators. By multiplying the operator from both left and right by powers, it also suffices to consider the case $\alpha = \beta = 0$ with the assumption on the index sets transformed to

$$\inf E_{\mathrm{rb}}, \inf E_{\mathrm{lb}} > 0.$$

If the kernel vanishes at $\mathrm{bf}(X_b^2)$ then Lemmas 5.32 and 5.33 apply. Thus it can be assumed that the kernel has support near bf. It is also convenient to divide it again into two parts, one with support away from rb and the other with support away from lb. In fact it suffices to consider the first one of these since the other is the adjoint of such an operator so will also satisfy (5.103). Again using Lemma 5.32 the kernel can be supposed to have support near bf and u in (5.103) can be supposed to have its support in $x < \epsilon$. Then the representation (4.68) can be used:

$$\langle Ru, u\rangle = \int_\delta^\infty \int_0^\epsilon \int_{\partial X \times \partial X} u(x, y)\kappa(x, s, y, y')u(x/s, y')\frac{ds}{s}\frac{dx}{x}dydy'.$$

This can be estimated by the Cauchy-Schwarz inequality, using some small $r > 0$:

$$\begin{aligned} |\langle Ru, u\rangle|^2 \le &\int_\delta^\infty \int_0^\epsilon \int_{\partial X \times \partial X} |s^{-r}\kappa(x, s, y, y')u(x, y)|^2\frac{ds}{s}\frac{dx}{x}dydy' \\ &\times \int_\delta^\infty \int_0^\epsilon \int_{\partial X \times \partial X} |s^r u(x/s, y')|^2\frac{ds}{s}\frac{dx}{x}dydy'. \end{aligned}$$

Both integrals on the right are bounded by the square of the L^2 norm, so the theorem is proved.

EXERCISE 5.35. Go through the construction of B in (5.102). The symbol of B should be $c_0 = (C - \sigma(A)^2)^{\frac{1}{2}}$. Check that if C is large enough this is well-defined as a symbol modulo $S^{-\infty}$. Find a self-adjoint operator B_0 with symbol c_0 and check that

$$R_1 = B_0^2 + A^*A - C \in \Psi_b^{-1}(X; {}^b\Omega^{\frac{1}{2}}).$$

Next look for $B_{(1)} = B_0 + B_1$ with B_1 self-adjoint and or order -1. Deduce that the symbol of B_1 should satisfy

$$2c_0\sigma_{-1}(B_1) = \sigma_{-1}(R_1).$$

Show that this has a self-adjoint solution and that the remainder term

$$R_2 = B_1^2 + A^*A - C$$

is of order -2. Now give a similar argument which serves as the inductive step to show that for every $k \in \mathbb{N}$ there exists a self-adjoint B_k of order $-k$ such that

$$R_{k+1} = \left[\sum_{j=0}^{k} B_j\right]^2 + A^*A - C \in \Psi_b^{-k-1}(X; {}^b\Omega^{\frac{1}{2}}).$$

Finally use asymptotic summation to construct B satisfying (5.102), making sure that it is self-adjoint.

5.16. Calculus with bounds.

Although a considerable effort is expended below to keep rather detailed information on the asymptotic expansions of kernels, it is convenient for various purposes to have available operators satisfying only *conormal bounds.* This is quite analogous, for the usual calculus of pseudodifferential operators, to the relationship between the general symbol estimates (4.42) and the polyhomogeneous symbols in (4.43). Although the kernels will only satisfy conormal bounds at lb and rb, some smoothness up to $\mathrm{bf}(X_b^2)$ will be maintained.

Let $\mathcal{W}$ be the space of $\mathcal{C}^\infty$ vector fields on X_b^2 which are tangent to lb and rb but may be transversal to bf. Choose a non-vanishing section $\nu \in \mathcal{C}^\infty(X_b^2; {}^b\Omega^{\frac{1}{2}})$ and then consider

(5.104)
$$\left\{K \in \rho_{\mathrm{bf}}^{-\frac{1}{2}} H_b^\infty(X_b^2; {}^b\Omega^{\frac{1}{2}}); K = K'\nu,\ \mathcal{W}^p K' \subset \rho_{\mathrm{bf}}^{-\frac{1}{2}} H_b^\infty(X_b^2)\ \forall\ p \in \mathbb{N}\right\}.$$

The factor of $\rho_{\mathrm{bf}}^{-\frac{1}{2}}$ is included so as not to force any vanishing of the elements at $\mathrm{bf}(X_b^2)$. Away from bf, (5.104) involves no further regularity than $H_b^\infty(X_b^2; {}^b\Omega^{\frac{1}{2}})$. Taking a product neighbourhood of bf with r a defining function for it, the regularity in (5.104) just requires that $K = K(r, \tau, y, y')|\frac{dr}{r}|^{\frac{1}{2}}$, with

$$K(r, \cdot) \in \mathcal{C}^\infty([0, \epsilon]; H_b^\infty([-1, 1] \times (\partial X)^2; {}^b\Omega^{\frac{1}{2}})). \tag{5.105}$$

EXERCISE 5.36. Show that (5.104) is equivalent to (5.105) together with the condition that $\phi K \in H_b^\infty(X_b^2; {}^b\Omega^{\frac{1}{2}})$ if $\phi \in \mathcal{C}^\infty(X_b^2)$ has support disjoint from $\mathrm{bf}(X_b^2)$.